Integral points of bounded height on partial equivariant compactifications of vector groups

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Abstract. — We establish asymptotic formulas for the number of integral points of bounded height on partial equivariant compactifications of vector groups.

Résumé. — Nous établissons un développement asymptotique du nombre de points entiers de hauteur bornée dans les compactifications équivariantes partielles de groupes vectoriels et de tores algébriques.

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Contents

1. Introduction

In this paper we study the distribution of integral points of bounded height on partial equivariant compactifications of additive groups, *i.e.*, quasi-projective algebraic varieties defined over number fields, equipped with an action of a vector group $G \simeq \mathbb{G}_a^n$, and containing G as an open dense orbit.

The case of projective compactifications has been the subject of [3]. Motivated by a conjecture of Manin, we established analytic properties of corresponding height zeta functions and deduced asymptotic formulas for the number of *rational* points of bounded height. Examples of such varieties are projective spaces of any dimension and their blow-ups along a reduced subscheme contained in a hyperplane, as well as certain singular Del Pezzo surfaces (listed in [5]).

The study of *integral* solutions of diophantine equations is a major branch of number theory. Our aim was to extend to the theory of *integral* points the geometric framework and techniques developed in the context of Manin's conjecture.

Important ingredients in the context of rational points on a smooth projective variety X over a number field F are:

– the convex geometry of the cone of effective divisors in the Picard group of X;
– the positivity of the anticanonical class of X, which is assumed to be ample (X Fano), or at least big;
– Tamagawa measures on the adelic space $X(\mathbb{A}_F)$ of X.

Manin's conjecture [6], as refined by Peyre [8] and [1, 2], then predicts that the number of F-rational points of anticanonical height $\leqslant B$ is asymptotic to

$$\alpha(X)\beta(X)\tau(X)B(\log B)^{b-1},$$

where $b = \operatorname{rank}\operatorname{NS}(X)$, $\alpha(X)$ is a rational number depending on the location of the anticanonical class in the effective cone, $\beta(X)$ is the cardinality of the Galois cohomology group $\mathrm{H}^1(\operatorname{Pic}(\bar{X}))$, and $\tau(X)$ is the volume of the closure of $X(F)$ in $X(\mathbb{A}_F)$ with respect to the Tamagawa measure derived from the metrization of the canonical line bundle defining the height.

In [4], we generalized the theory of Tamagawa measures to the quasi-projective case and established asymptotic formulas for volumes of analytic and adelic height balls of growing radius.

In the present paper, we establish analytic properties of height zeta functions for integral points on partial equivariant compactifications U of vector groups. We assume that U is the complement to a G-invariant divisor D in a smooth projective equivariant compactification X of G, and that, geometrically, D has simple normal crossings. We fix integral models for X (assumed to proper over $\operatorname{Spec}(\mathfrak{o}_F)$), D and U, and consider the subset of S-integral points in $X(F)$ with respect to these models, S being a finite set of places of F, including all archimedean places. This will be denoted by $\mathscr{U}(\mathfrak{o}_{F,S})$. The case of rational points corresponds to the case $D = \varnothing$; then $\mathscr{U}(\mathfrak{o}_{F,S}) = X(F)$ provided the chosen model of D is also empty.

In this setup we defined in [4], for each place $v \in S$, a simplicial complex $\mathscr{C}^{\mathrm{an}}_{F_v}(D)$, which we called the *analytic Clemens complex*, and which encodes the incidence properties of the v-adic manifolds given by the irreducible components of D. Its dimension is equal to the maximal number of irreducible components of D defined over the local field F_v whose intersection has F_v-points, minus one.

By definition, the log-canonical line bundle on X is $K_X + D$. The log-anticanonical line bundle is its opposite; in the case under study, it is big. An adelic metrization of this line bundle defines a height function H on $X(F)$ which satisfies a finiteness property: for any real number B, the set of points x in $G(F)$ such that $H(x) \leqslant B$ is finite. The formula proved in Theorem 3.5.6 asserts that the number $N(B)$ of points in $G(F) \cap \mathscr{U}(\mathfrak{o}_{F,S})$ of bounded log-anticanonical height $\leqslant B$ satisfies

$$N(B) \sim \Theta B (\log B)^{b-1},$$

where

$$b = \operatorname{rank}(\operatorname{Pic}(U)) + \sum_{v \in S} (1 + \dim \mathscr{C}^{\mathrm{an}}_{F_v}(D))$$

and Θ is a positive real number: a product of an adelic volume, involving places outside S, and contributions from places in S given by sums of integrals over the minimal dimensional strata of the analytic Clemens complex.

The notions of S-integral points and heights extend to the adelic group $G(\mathbb{A}_F)$. In a very general context, we had proved in [4] an asymptotic formula for the volume $V(B)$ of S-integral adelic points in $G(\mathbb{A}_F)$ of log-anticanonical height bounded by B. Our main result says that, for $B \to \infty$,

$$N(B) \sim V(B),$$

see Remark 3.5.7.

The explicit description of the constant Θ as a product of an adelic volume, and of local volumes, implies an equidistribution property of S-integral points for some explicit measure on $X(\mathbb{A}_F)$. (See Section 3.5.8.) This measure is a regularized measure on the adelic space $U(\mathbb{A}_F^S)$. At places $v \in S$, it is localized on the strata of minimal dimension of the F_v-analytic Clemens complex.

The proof relies on a strategy developed in our previous papers on rational points. We introduce the height zeta function

$$\mathrm{Z}(s) = \sum_{x \in G(F) \cap \mathscr{U}(\mathfrak{o}_{F,S})} H(x)^{-s},$$

for a complex parameter s. We prove its convergence for $\mathrm{Re}(s) > 1$, and then establish its meromorphic continuation in some half-plane $\mathrm{Re}(s) > 1 - \delta$ whose poles are located on finitely many arithmetic progression on the line $\mathrm{Re}(s) = 1$. The pole of highest order, equal to b, is at $s = 1$. The asymptotic formula for $N(B)$ follows from an appropriate Tauberian theorem.

Observe that the restriction of the height H to $G(F)$ can be written as a product over all places v of F of a local height function H_v defined on $G(F_v)$. Hence, we can extend

the height as a function on $G(\mathbb{A}_F)$. Let us also introduce, for each place $v \notin S$, the characteristic function δ_v of the set $\mathscr{U}(\mathfrak{o}_v)$ of v-integral points in $X(F_v)$ and set $\delta_v \equiv 1$ for $v \in S$. Using the Poisson summation formula for the locally compact group $G(\mathbb{A}_F)$ and its cocompact discrete subgroup $G(F)$, we can write

$$\mathrm{Z}(s) = \sum_{x \in G(F)} \prod_{v \in \mathrm{Val}(F)} \delta_v(x) H_v(x)^{-s} = \sum_{\psi \in G(\mathbb{A}_F)/G(F))^*} \prod_v \mathscr{F}_v(\delta_v H_v^{-s}; \psi_v),$$

where, for each place v of F,

$$\mathscr{F}_v(\delta_v H_v^{-s}; \psi_v) = \int_{G(F_v)} \delta_v(x) H_v(x)^{-s} \psi_v(x) \, \mathrm{d}x_v$$

is the local Fourier transform at the local component ψ_v of the global character ψ.

The Fourier transform at the trivial character furnishes the main term in the right hand side of the Poisson formula. Its analytic properties have been established in [4].

In the case of rational points, all the analogous local integrals $\mathscr{F}_v(H_v^{-s}; \psi_v)$ converge absolutely when s belongs to the half-plane $\{\mathrm{Re}(s) > 0\}$ and the poles of the global Fourier transform are all accounted for by the Euler products. The main technical difficulty which appears for integral points is that we need to establish, for places $v \in S$, some meromorphic continuation to the left of $\mathrm{Re}(s) = 1$ of the local integrals $\mathscr{F}_v(\delta_v H_v^{-s}; \psi_v)$. Moreover, we need to provide decay estimates sufficient to ensure the convergence of the right hand side in some half-plane $\mathrm{Re}(s) > 1 - \delta$.

For places in S, the local integrals at the trivial character are integrals of Igusa-type. Such integrals were investigated in our previous paper [4]: using Tauberian theorems, the asymptotic of local height balls was deduced from their analytic properties. For non-trivial characters, we are led to consider *oscillatory* Igusa-type integrals, higher-dimensional analogs of integrals of the form

$$\int_{F_v} |x|^{s-1} \, \psi(ax^d) \Phi(x) \, \mathrm{d}x,$$

where $d \in \mathbf{Z}$, $a \in F_v$, and Φ is a Schwartz function on F_v. Establishing their meromorphic continuation, with appropriate decay in the parameter ψ, is the technical heart of this work. This is accomplished in Section 3.4, using estimates for stationary phase integrals from Section 2.

Ultimately, the corresponding Euler products at non-trivial characters do not contribute to the main pole of the height zeta function. However, we found that this property was more subtle than in the case of rational points (see Section 3.5.3).

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2. One-dimensional theory

2.1. Local theory: characters and quasi-characters

Let F be a local field of characteristic zero, *i.e.*, a completion of a number field with respect to a valuation. It is locally compact and we denote by μ a Haar measure on F. This defines an absolute valuation on F, normalized by the equality $\mu(a\Omega) = |a|\,\mu(\Omega)$. We write Tr and N for the absolute trace and norm, these take values in the corresponding completion of $\mathbf{Q}$. For non-archimedean F, let $\mathfrak{o}_F$ be the ring of integers, $\mathfrak{o}_F^*$ its multiplicative group, $\mathfrak{d}_F$ the local different of F, and q the order of the residue field. For archimedean F, we write $\mathfrak{o}_F^*$ for the subgroup of elements of absolute value one. From now on we fix a local Haar measure μ on F, normalizing so that

$$\int_{|x|\leqslant 1} \mathrm{d}\mu(x) = \begin{cases} 2 & \text{if } F \text{ is real;} \\ 2\pi & \text{if } F \text{ is complex;} \\ \mathrm{N}(\mathfrak{d}_F)^{-1/2} & \text{if } F \text{ is non-archimedean.} \end{cases}$$

We will often write $\mathrm{d}x$ instead of $\mathrm{d}\mu(x)$.

Let ψ be the unimodular character of F with respect to $\mathrm{d}x$. Concretely,

$$\psi(x) = \begin{cases} e^{2\pi\mathrm{i}\,\mathrm{Tr}(x)} & \text{for non-archimedean fields } F, \\ e^{-2\pi\mathrm{i}\,\mathrm{Tr}(x)} & \text{otherwise.} \end{cases}$$

(For non-archimedean fields, the exponential is defined by embedding naturally $\mathbf{Q}_p/\mathbf{Z}_p$ into $\mathbf{Q}/\mathbf{Z}$ and then applying $e^{2\pi\mathrm{i}\cdot}$.) The map

$$F \times F \to \mathbf{C}^*, \quad (a, x) \mapsto \psi(ax)$$

is a perfect pairing. The local Fourier transform is the function on F defined by

$$\hat{f}(a) = \int_F f(x)\psi(ax)\mathrm{d}x,$$

whenever the integral converges.

Let $\mathscr{Q}(F^*)$ be the set of quasi-characters of F^*, *i.e.*, continuous homomorphisms $\chi\colon F^* \to \mathbf{C}^*$. Examples are given by the *unramified characters, i.e.*, those whose restriction to $\mathfrak{o}_F^*$ is trivial; they are of the form $x \mapsto |x|^s$, for some complex number $s \in \mathbf{C}$.

The image of the absolute value is equal to $\mathbf{R}_{>0}$ if F is archimedean, and to $q^{\mathbf{Z}}$ if F is non-archimedean; moreover, the morphism of groups $|\cdot| : F^* \to |F^*|$ has a section, given by $t \mapsto t$ if $F = \mathbf{R}$, $t \mapsto \sqrt{t}$ if $F = \mathbf{C}$, and $q \mapsto \varpi^{-1}$, where ϖ is any fixed uniformizing element when F is non-archimedean. Using this section, we identify the group F^* with the product $|F^*| \times \mathfrak{o}_F^*$, and write any quasi-character χ as

$$\chi(x) = |x|^s\,\tilde{\chi}(\tilde{x}), \qquad x = (|x|\,, \tilde{x}).$$

The complex number $s = s(\chi)$ is determined uniquely by χ if F is archimedean, and uniquely modulo $2\pi\mathrm{i}/\log(q)$, if F is non-archimedean. As in [9], we call the *exponent* of χ the real number $r = r(\chi) = \mathrm{Re}(s(\chi))$. A quasi-character is a character of F^* if and only if its exponent is zero.

We say that two quasi-characters are *equivalent* if their quotient is unramified. By the map $\chi \mapsto s(\chi)$, the set of equivalence classes of quasi-characters is viewed as the Riemann surface $\mathbf{C}$ (if F is archimedean) or $\mathbf{C}/(2\pi\mathrm{i}/\log(q))\mathbf{Z})$ (if F is non-archimedean). The space $\mathscr{Q}(F^*)$ is a trivial bundle over this surface, with discrete topology on the fibers; we endow it with the corresponding structure of a Riemann surface,

Let us assume that F is non-archimedean. Let n be the minimal natural number such that χ is trivial on the subgroup $(1+\varpi^n\mathfrak{o}_F)$ of F^*. The $\mathfrak{o}_F$-ideal generated by ϖ^n is called the *conductor* of χ.

2.2. Local Tate integrals

We now fix a Haar measure on F^*:

$$\mathrm{d}^\times x = \begin{cases} \frac{\mathrm{d}x}{|x|} & \text{if } F \text{ is archimedean} \\ (1-\frac{1}{q})^{-1}\frac{\mathrm{d}x}{|x|} & \text{if } F \text{ is non-archimedean} \end{cases}$$

When F is non-archimedean, this normalization implies

$$\int_{\mathfrak{o}_F^*} \mathrm{d}^\times x = \mathrm{N}(\mathfrak{d}_F)^{-1/2}.$$

Let $\mathscr{T}(F)$ be the class of functions $\Phi\colon F \to \mathbf{C}$ satisfying the following properties:

– Φ and $\hat{\Phi}$ are continuous and absolutely integrable over F with respect to $\mathrm{d}x$;

– for all $r > 0$ the functions $x \mapsto \Phi(x)\,|x|^r$ and $x \mapsto \hat{\Phi}(x)\,|x|^r$ are absolutely integrable over F^* with respect to $\mathrm{d}^\times x$.

For $\Phi \in \mathscr{T}(F)$ and χ with $r(\chi) > 0$ the integral

$$\zeta(\Phi,\chi) = \int_F \Phi(x)\chi(x)\mathrm{d}^\times x,$$

is absolutely convergent. Moreover, it defines a holomorphic function on the set of equivalence classes of quasi-characters with $r(\chi) > 0$. This function is called the local ζ-function.

As a classical example of such integrals, we define, for any complex number s such that $\mathrm{Re}(s) > 0$,

$$\zeta_F(s) = \zeta(\mathbf{1}_{\mathfrak{o}_F}, |\cdot|^s) = \int_{|x|\leqslant 1} |x|^s \,\mathrm{d}^\times x.$$

Explicitly (see [9], p. 344):

$$\zeta_F(s) = \begin{cases} 2/s & \text{if } F = \mathbf{R}; \\ 2\pi/s & \text{if } F = \mathbf{C}; \\ \mathrm{N}(\mathfrak{d}_F)^{-1/2}/(1-q^{-s}) & \text{if } F \text{ is non-archimedean.} \end{cases}$$

The corresponding residue is denoted by

$$\mathrm{c}_F = \lim_{s\to 0} s \int_{|x|\leqslant 1} |x|^s \,\mathrm{d}^\times x = \begin{cases} 2 & \text{if } F = \mathbf{R}; \\ 2\pi & \text{if } F = \mathbf{C}; \\ \mathrm{N}(\mathfrak{d}_F)^{-1/2}/\log q & \text{otherwise.} \end{cases}$$

These constants will appear in the definition of residue measures below.

The main theorem of the local theory in [9, Theorem 2.4.1] is:

PROPOSITION 2.2.1. — *There exists a meromorphic function ρ on $\mathscr{Q}(F^*)$ such that for any function $\Phi \in \mathscr{T}(F)$, the local ζ-function has an meromorphic continuation to the domain of all quasi-characters given by the functional equation*

$$\zeta(\Phi,\chi) = \rho(\chi)\zeta(\hat{\Phi},\hat{\chi})$$

where $\hat{\chi}(x) = |x|\,\chi(x)^{-1}$.

We recall that in [9], the function ρ is defined for exponents $r(\chi) \in (0,1)$ by the functional equation (it is holomorphic in that domain) and extended by meromorphic continuation for all quasi-characters. It is also computed explicitly in that paper. As a consequence, we obtain the following corollary.

COROLLARY 2.2.2. — *The local ζ-function is holomorphic at any character different from the trivial character. Moreover, if χ is the quasi-character $x \mapsto |x|^s$, for $s \to 0$, then*

$$\Phi(0) = \lim_{s\to 0} \zeta(\Phi, |\cdot|^s)/\zeta_F(s).$$

2.3. Oscillatory integrals

Let F be a local field. In this section we study integrals of the form

$$\mathscr{I}(\Phi,a,d,\chi) = \int_F \psi(ax^d)\chi(x)\Phi(x)\mathrm{d}x, \quad d\in\mathbf{Z}, \quad a\in F, \quad \chi\in\mathscr{Q}(F^*),$$

for suitable test functions Φ on F. When d is nonnegative, the behavior of such integrals as a function of χ is explained by Proposition 2.2.1. We need to understand the decay of such integrals when the absolute value of the parameter a grows to infinity, for positive integers d. This result will also be used to establish a meromorphic continuation of these integrals, with respect to χ, when d is negative.

When the test function Φ vanishes in a neighborhood of 0 and belongs to some Schwartz class, the integral $\mathscr{I}$ is a *non-stationary phase* integral and its decay with respect to the parameter a is classical. For example, the integral

$$\int_{\mathfrak{o}_F^*} \chi(x)\psi(ax)\,\mathrm{d}^\times x,$$

can be computed explicitly (see, e.g., [11, p. 20–21]); it vanishes for $|a|$ large enough with respect to the conductor $\mathfrak{f}(\chi)$. We will need the following version of this computation.

Lemma 2.3.1. — *Let us assume that F is ultrametric and let ϖ be a uniformizing element. Set $c = \log_q \#(\mathfrak{o}_F/(d\mathfrak{d}_F))$. For any $a \in F$, any $n \in \mathbf{Z}$ such that $q^{n+c} < |a| \leqslant q^{2n}$, and any $\xi \in \mathfrak{o}_F \setminus (\varpi)$, we have*

$$\int_{\xi+\varpi^n\mathfrak{o}_F} \psi(ax^d)\,\mathrm{d}x = 0.$$

Proof. — Let I be this integral. Making the change of variables $x = \xi(1+\varpi^n u)$, we obtain

$$I = q^{-n} \int_{\mathfrak{o}_F} \psi(a\xi^d(1+\varpi^n u)^d)\,\mathrm{d}u.$$

Without loss of generality, we can also assume $\xi = 1$, replacing a by $a\xi^d$. Now,

$$a(1+\varpi^n u)^d = a + \binom{d}{1} a\varpi^n u + \binom{d}{2} a\varpi^{2n}u^2 + \cdots + \binom{d}{d} a\varpi^{dn}u^d.$$

If $a\varpi^{2n}$ belongs to $\mathfrak{o}_F$, all terms from the third one on are elements of $\mathfrak{o}_F$, so that

$$\psi(a(1+\varpi^n u)^d) = \psi(a)\psi(da\varpi^n u)$$

for any $u \in \mathfrak{o}_F$. In that case,

$$I = q^{-n}\psi(a) \int_{\mathfrak{o}_F} \psi(da\varpi^n u)\,\mathrm{d}u.$$

This is the integral of an additive character of $\mathfrak{o}_F$, so vanishes if and only if this character is non-trivial. By definition of the different, this happens precisely if $da\varpi^n \notin \mathfrak{d}_F^{-1}$. Consequently, $I = 0$ when $|a| \leqslant q^{2n}$ and $|a| > q^n/|d\mathfrak{d}_F| = q^{n+c}$. □

We recall some terminology. A complex valued function on F is called *smooth* if it is infinitely differentiable, or locally constant, according to whether F is archimedean or not. *Schwartz functions* are smooth functions with compact support. The vector space $\mathscr{S}(F)$ of Schwartz functions has a natural topology; a subset of $\mathscr{S}(F)$ is bounded when:

a) the supports of all of its elements are contained in a common compact subset;

b) if $F = \mathbf{R}$, then for each integer n, their nth derivatives are uniformly bounded;

c) if $F = \mathbf{C}$, then for all $(m,n) \in \mathbf{N}^2$, their partial derivatives $\partial_x^m \partial_y^n$ are uniformly bounded;

d) if F is ultrametric, then there exists a positive real number δ such that all of its elements are constant on any ball of radius δ in F.

Proposition 2.3.2. — *Let F be a local field and Φ a Schwartz function on F. Let d be a positive integer. For any complex number s such that $\mathrm{Re}(s) > 0$, let*

$$\kappa(s) = \min\big(\frac{1}{2}, \frac{\mathrm{Re}(s)}{d}\big).$$

Then, as as $|a| \to \infty$ and $\mathrm{Re}(s) > 0$, we have

$$\mathscr{I}(\Phi, a, d, s) = \int_F |x|^{s-1}\,\psi(ax^d)\Phi(x)\,\mathrm{d}x \ll \zeta_F(\mathrm{Re}(s)) \min\big(1, |a|^{-\kappa(s)}\big).$$

The bound is uniform when Φ belongs to a fixed bounded subset of the space of Schwartz functions and $\operatorname{Re}(s)>0$ *is bounded from above.*

REMARK 2.3.3. — In fact, except when $F=\mathbf{C}$, we prove the proposition with $\kappa(s)$ replaced by $\operatorname{Re}(s)/d$.

Proof. — To simplify, we put $I(a)=\mathscr{I}(\Phi,a,d,s)$. We may assume that Φ is real-valued. By a change of variables, we also assume that Φ is zero outside of the unit ball.

The case $F=\mathbf{R}$. We have $\psi(x)=\exp(-2\pi\mathrm{i}x)$. We introduce a parameter $\varepsilon>0$ and split the integral:

$$I(a)=\int_{-\varepsilon}^{\varepsilon}|x|^{s-1}\psi(ax^d)\Phi(x)\,\mathrm{d}x+\int_{|x|\geqslant\varepsilon}|x|^{s-1}\psi(ax^d)\Phi(x)\,\mathrm{d}x.$$

The first integral is bounded from above as

$$\left|\int_{-\varepsilon}^{\varepsilon}|x|^{s-1}\psi(ax^d)\Phi(x)\,\mathrm{d}x\right|\leqslant\varepsilon^{\operatorname{Re}(s)}\left\|\Phi\right\|_\infty\zeta_{\mathbf{R}}(\operatorname{Re}(s)).$$

Integration by parts in the second integral yields

$$\begin{aligned}\int_{\varepsilon}^{\infty}x^{s-1}\psi(ax^d)\Phi(x)\,\mathrm{d}x&=\int_{\varepsilon}^{\infty}x^{s-d}\Phi(x)\,x^{d-1}\psi(ax^d)\,\mathrm{d}x\\&=\left[\frac{-1}{2\pi\mathrm{i}da}x^{s-d}\Phi(x)\psi(ax^d)\right]_{\varepsilon}^{\infty}\\&\qquad+\frac{1}{2\pi\mathrm{i}da}\int_{\varepsilon}^{\infty}x^{s-d-1}\left((s-d)\Phi(x)+x\Phi'(x)\right)\psi(ax^d)\mathrm{d}x.\end{aligned}$$

Consequently, its absolute value is bounded from above by

$$\begin{aligned}&\frac{1}{2\pi d\,|a|}\varepsilon^{\operatorname{Re}(s)-d}\left\|\Phi\right\|_\infty+\frac{1}{2\pi d\,|a|}\int_{\varepsilon}^{1}x^{\operatorname{Re}(s)-d-1}\left(|\operatorname{Re}(s)-d|\,|\Phi|+x\,|\Phi'|\right)\,\mathrm{d}x\\&\qquad\qquad\ll(2\pi d\,|a|)^{-1}\left(\varepsilon^{s-d}\left\|\Phi\right\|_\infty+\varepsilon^{s-d}(\left\|\Phi\right\|_\infty+\left\|\Phi'\right\|_\infty)\right).\end{aligned}$$

We have a similar upper-bound for the integral from $-\infty$ to $-\varepsilon$.

Fix ε so that $\varepsilon^d\,|a|=1$. Adding the obtained estimates, we have

$$|I(a)|\ll|a|^{-\operatorname{Re}(s)/d}\,\zeta_{\mathbf{R}}(\operatorname{Re}(s))\left(\left\|\Phi\right\|_\infty+\left\|\Phi'\right\|_\infty\right),$$

where the constant understood under $\ll$ is absolute.

The case $F=\mathbf{C}$. Note that in this case, the absolute value $|\cdot|$ is the square of the usual absolute value, but we preferred using the usual one in our proof. We will thus need to obtain an exponent $2\kappa(s)=\min(1,2\operatorname{Re}(s)/d)$. Recall that

$$\psi(u)=\exp(-2\mathrm{i}\pi\operatorname{Re}(u))\qquad\text{for}\qquad u\in\mathbf{C};$$

we write $a=\omega\exp(\mathrm{i}\alpha)$ with $\omega=|\alpha|$ and $\alpha\in\mathbf{R}$. Similarly, put $z=r\exp(\mathrm{i}\theta)$, so that

$$\psi(az^d)=\exp\big(-2\mathrm{i}\pi\omega r^d\cos(d\theta+\alpha)\big).$$

Using polar coordinates, we have

$$I(a) = \int_0^1 \mathrm{d}\theta \int_0^\infty \exp(2\mathrm{i}\pi\omega r^d \cos(d\theta+\alpha))\Phi(r\exp(\mathrm{i}\theta))\, r\,\mathrm{d}r.$$

We write $I(a;\theta)$ for the inner integral; the result we proved for $F=\mathbf{R}$ implies that

$$|I(a;\theta)| \ll \omega^{-2\operatorname{Re}(s)/d} \left|\cos(d\theta+\alpha)\right|^{-2\operatorname{Re}(s)/d};$$

moreover, a trivial inequality $|I(a;\theta)| \ll 1$ holds. We now integrate the better of these upper bounds over $\theta\in[0;2\pi]$. The integral over θ such that $|\cos(d\theta+\alpha)|\leqslant 1/\omega$ is $\ll 1/\omega$. The angles θ such that $|\cos(d\theta+\alpha)|\leqslant 1/\omega$ form a union of intervals of lengths $\approx 1/\omega$; the integral over these will be $\ll 1/\omega$. When $\omega\to\infty$, the integral over the remaining angles grows as $\omega^{-2\operatorname{Re}(s)/d}\int_{1/\omega}^1 u^{-2\operatorname{Re}(s)/d}\,\mathrm{d}u$, that is:

$$\omega^{-2\operatorname{Re}(s)/d}\max(1,\omega^{-1+2\operatorname{Re}(s)/d}) = \max(\omega^{-2\operatorname{Re}(s)/d},\omega^{-1}).$$

Finally,

$$I(a) \ll |a|^{-\min(1,2\operatorname{Re}(s)/d)},$$

as claimed.

The case when F is non-archimedean. Let ϖ be a generator of the maximal ideal of $\mathfrak{o}_F$ and let $q=|\varpi|^{-1}=\#(\mathfrak{o}_F/(\varpi))$. As above, we assume that Φ is real-valued and that its support is contained in $\mathfrak{o}_F$. Let $n(\Phi)$ be the least positive integer such that Φ is constant on residue classes modulo ϖ^n. For any nonnegative integer k, let Φ_k be the Schwartz function on $\mathfrak{o}_F$ defined by $\Phi_k(u)=\Phi(\varpi^k u)$; one has $n(\Phi_k)=\max(n(\Phi)-k,1)$. For some nonnegative integer K (to be chosen later), we write

$$\begin{aligned} I(a) &= \int_F |x|^{s-1}\psi(ax^d)\Phi(x)\,\mathrm{d}x \\ &= \sum_{k=0}^{K-1}\int_{(\varpi^k)\setminus(\varpi^{k+1})}\psi(ax^d)\Phi(x)\,\mathrm{d}x + \int_{(\varpi^K)}|x|^{s-1}\psi(ax^d)\Phi(x)\,\mathrm{d}x \\ &= \sum_{k=0}^{K-1} q^{-sk}\int_{\mathfrak{o}_F\setminus(\varpi)}\psi(a\varpi^{kd}x^d)\Phi_k(x)\,\mathrm{d}x + q^{-sK}\int_{\mathfrak{o}_F}|x|^{s-1}\psi(a\varpi^{Kd}x^d)\Phi_K(x)\,\mathrm{d}x. \end{aligned}$$

Let m be such that $|a|=q^m$. By Lemma 2.3.5 below, the term corresponding to the index k vanishes if

$$\left|a\varpi^{kd}\right| \geqslant \max(q^{2c+2}, q^{n(\Phi_k)+c+1}),$$

that is if

$$m-kd \geqslant \max(2c+2, n(\Phi)-k+c+1, c+2) = \max(2c+2, n(\Phi)+c+1-k).$$

Since $d\geqslant 1$, the right hand side decreases slower than left hand side, and this holds for all $k\in\{0,\dots,K-1\}$ if it holds for $k=K-1$, that is if

$$m-(K-1)d \geqslant 2c+2 \quad\text{and}\quad m-(K-1)d \geqslant n(\Phi)+c+1-(K-1),$$

in other words

$$(K-1)d \leqslant m-2c-2 \quad \text{and} \quad (K-1)(d-1) \leqslant m-n(\Phi)-c-1.$$

We choose K to be the largest integer satisfying these two inequalities, namely

$$K = 1+\min\left(\left\lfloor\frac{m-2c-2}{d}\right\rfloor, \left\lfloor\frac{m-n(\Phi)-c-1}{d-1}\right\rfloor\right).$$

This integer is positive when $m \geqslant \min(2c+2, n(\Phi)+c+1)$. In that case,

$$I(a) = q^{-sK}\int_{\mathfrak{o}_F} \psi(a\varpi^{Kd})\,|x|^{s-1}\,\Phi_k(x)\,\mathrm{d}x$$

so that

$$|I(a)| \leqslant q^{-K\operatorname{Re}(s)}\,\|\Phi\|_\infty \int_{\mathfrak{o}_F} |x|^{\operatorname{Re}(s)-1}\,\mathrm{d}x \leqslant q^{-K\operatorname{Re}(s)}\,\|\Phi\|_\infty\,\zeta_F(\operatorname{Re}(s)),$$

hence

$$|I(a)| \leqslant \zeta_F(\operatorname{Re}(s))\,\|\Phi\|_\infty \max\left(q^{2(c+1)/d}\,|a|^{-1/d}, q^{(n(\Phi)+c+1)/(d-1)}\,|a|^{-1/(d-1)}\right).$$

As a consequence, when $|a| \geqslant 1$, we find

$$|I(a)| \leqslant \zeta_F(\operatorname{Re}(s))\,\|\Phi\|_\infty \max(q^{2(c+1)/d}, q^{(n(\Phi)+c+1)/(d-1)})\,|a|^{-1/d}.$$

Since $|I(a)| \leqslant \zeta_F(\operatorname{Re}(s))\,\|\Phi\|_\infty$ for any $a \in F$, we conclude that

$$|I(a)| \leqslant \zeta_F(\operatorname{Re}(s))\,\|\Phi\|_\infty \max(q^{2(c+1)/d}, q^{(n(\Phi)+c+1)/(d-1)}) \max(1, |a|)^{-1/d}. \tag{2.3.4}$$

□

We now prove the Lemma used in the non-archimedean case.

Lemma 2.3.5. — *Let Φ be a Schwartz function on F with support in $\mathfrak{o}_F$. Let $n(\Phi)$ be the least positive integer n such that Φ is constant on residue classes modulo ϖ^n. For any $a \in F$ such that*

$$|a| \geqslant \max(q^{n(\Phi)+c+1}, q^{2(c+1)}),$$

one has

$$\int_{\mathfrak{o}_F\setminus(\varpi)} \Phi(x)\psi(ax^d)\,\mathrm{d}x = 0.$$

Proof. — Let n be any integer such that $n \geqslant n(\Phi)$; we can write

$$\int_{\mathfrak{o}_F\setminus(\varpi)} \Phi(x)\psi(ax^d)\,\mathrm{d}x = \sum_{\substack{\xi\neq 0 \\ (\mathrm{mod}\ (\varpi^n))}} \Phi(\xi)\int_{\xi+\varpi^n\mathfrak{o}_F} \psi(ax^d)\,\mathrm{d}x.$$

By Lemma 2.3.1, each term in this sum vanishes if $q^{n+c} < |a| \leqslant q^{2n}$. Let us set $|a| = q^m$; these conditions ($n+c+1 \leqslant m \leqslant 2n$ and $n \geqslant n(\Phi)$) are equivalent to ($\frac{1}{2}m \leqslant n \leqslant m-c-1$ and $n \geqslant n(\Phi)$). There exists such an integer n if and only if $n = m-c-1$ is a solution, *i.e.*, $m \geqslant 2(c+1)$ and $m \geqslant n(\Phi)+c+1$. □

We will need the following higher-dimensional generalization of Proposition 2.3.2.

Proposition 2.3.6. — *Let F be a local field and Φ a Schwartz function on F^n. Let $n \geqslant 1$ and let $d_1, \dots, d_n$ be positive integers. For $s_1, \dots, s_n \in \mathbf{C}$ set*

$$\kappa(s) = \min\bigl(1/2, \operatorname{Re}(s_1)/d_1, \dots, \operatorname{Re}(s_n)/d_n\bigr).$$

Assume that $\kappa(s) > 0$. Then, as $|a| \to \infty$,

$$\int_F |x_1|^{s_1-1} |x_2|^{s_2-1} \dots |x_n|^{s_n-1} \psi(ax_1^{d_1} \dots x_n^{d_n})\Phi(x)\, \mathrm{d}x \ll \min(1, |a|^{-\kappa(s)}) \prod_{j=1}^n \zeta_F(\operatorname{Re}(s_j)).$$

This bound is uniform when Φ ranges over a bounded subset of the space of Schwartz functions on F^n and all $\operatorname{Re}(s_j) \in \mathbf{R}_+^*$ are bounded from above.

Proof. — We may assume that Φ is real-valued and that its support is contained in the unit polydisk. For $j \in \{1, \dots, n\}$, set $\sigma_j = \operatorname{Re}(s_j)$. Let $I(a)$ be this integral. If $|a| \leqslant 1$, we bound the integral from above by the integral of its absolute value, replacing ψ by 1. Let us assume that $|a| \geqslant 1$. For $n = 1$, the claim follows from Lemma 2.3.2. By induction, we assume the result is known for $< n$ variables. Writing $s = (s_1, s')$, we obtain

$$I(a) \ll \prod_{j=2}^n \zeta_F(\sigma_j) \int_{|x_1| \leqslant 1} |x_1|^{\sigma_1 - 1} \max(1, \left|ax_1^{d_1}\right|^{-\kappa(s')})\, \mathrm{d}x_1.$$

We split this integral according to whether or not $|x_1| \leqslant |a|^{-1/d_1}$. Put $\sigma = \sigma_1$, $d = d_1$, and $\kappa = \kappa(s')$. We have

$$\int_{|x| \leqslant |a|^{-1/d}} |x|^{\sigma - 1}\, \mathrm{d}x = |a|^{-\sigma/d} \zeta_F(\sigma).$$

The second integral equals

$$|a|^{-\kappa} \int_{|a|^{-1/d} \leqslant |x| \leqslant 1} |x|^{\sigma - d\kappa - 1}\, \mathrm{d}x.$$

If $F = \mathbf{R}$ or $\mathbf{C}$, we use polar coordinates and obtain, up to the measure of $\mathfrak{o}_F^*$,

$$|a|^{-\kappa} \int_{|a|^{-1/d}}^1 r^{\sigma - d\kappa - 1}\, \mathrm{d}r = \frac{|a|^{-\kappa} - |a|^{-\sigma/d}}{\sigma - d\kappa} = \frac{1}{d} |a|^{-c},$$

for some $c \in (\kappa, \sigma/d)$. In particular,

$$c \geqslant \min(\kappa(s'), \frac{\sigma_1}{d_1}) = \min\bigl(\frac{\sigma_1}{d_1}, \dots, \frac{\sigma_n}{d_n}\bigr) = \kappa(s).$$

When F is non-archimedean, an analogous inequality holds, with the real integral replaced by a geometric series. Combining the inequalities, we have

$$I(a) \ll |a|^{-\sigma_1/d_1} \prod_{j=1}^n \zeta_F(\sigma_j) + |a|^{-\kappa(s)} \prod_{j=1}^n \zeta_F(\sigma_j) \ll \prod_{j=1}^n \zeta_F(\sigma_j) |a|^{-\kappa(s)},$$

as claimed. ∎

2.4. Igusa integrals with rapidly oscillating phase

PROPOSITION 2.4.1. — *Let* $\Phi\colon F\times\mathbf{C}\to\mathbf{C}$ *a function such that the functions* $s\mapsto\Phi(x,s)$ *are holomorphic for any* $x\in F$. *Assume that the functions* $x\mapsto\Phi(x,s)$ *belong to a bounded subset of the space of smooth compactly supported functions when* $\operatorname{Re}(s)$ *belongs to a fixed compact subset of* $\mathbf{R}$.

Let d *be a positive integer. For any* $a\in F^*$, *there exists an holomorphic function* $s\mapsto\eta_a(s)$ *defined for* $\operatorname{Re}(s)>-1$ *such that*

$$\eta_a(s)=\int_F |x|^{s-1}\,\psi(a/x^d)\Phi(x,s)\,\mathrm{d}x$$

for $\operatorname{Re}(s)>0$. *Moreover, when* $\operatorname{Re}(s)$ *belongs to a compact subset of* $(-1,+\infty)$, *it satisfies a uniform upper-bound of the form*

$$|\eta_a(s)|\ll |a|^{-1/d}.$$

Proof. — To give an unified proof, we introduce a version of "Littlewood–Paley" decomposition. If F is non-archimedean, let $\theta(x)=1$ if $|x|=1$ and 0 else; we then have

$$\sum_{n\in\mathbf{Z}}\theta(\varpi^n x)=1\quad\text{for all}\quad x\in F^*,$$

where ϖ is any uniformizing element of F, with $q=|\varpi|^{-1}$. If F is archimedean, let θ be any smooth nonnegative function such that:

- it is supported in the collar $1/2<|x|<2$;
- it equals 1 in a neighborhood of the collar $|x|=1$;
- the sum $\sum_{n\in\mathbf{Z}}\theta(2^nx)$ is a equal to 1 everywhere (except for $x=0$).

Such functions exist; for instance, take any function θ_1 satisfying the first two assumptions and which is positive on the collar $\frac{2}{3}<|x|<\frac{3}{2}$. Let $\Theta_1(x)=\sum_{n\in\mathbf{Z}}\theta_1(2^nx)$; it is positive everywhere and satisfies $\Theta(2x)=\Theta(x)$. Let $\theta(x)=\theta_1(x)/\Theta(x)$. It follows that for $x\neq 0$, $1=\sum_{n\in\mathbf{Z}}\theta(2^nx)$. By assumption, if $|x|\leqslant 1$, $\theta(2^nx)=0$ for $n\leqslant -1$, hence $\sum_{n\geqslant 0}\theta(2^nx)$ is 1 on the unit ball and is also compactly supported. We let $\varpi=q=2$ in this case.

Now, one has

$$\eta_a(s)=\int_F |x|^{s-1}\,\psi(a/x^d)\Phi(x,s)\sum_{n\in\mathbf{Z}}\theta(\varpi^n x)\,\mathrm{d}x=\sum_{n\in\mathbf{Z}}q^{-ns}\eta_{a,n}(s),$$

where

$$\eta_{a,n}(s)=\int_{q^{-1}<|x|<q}|x|^{s-1}\,\psi(aq^{dn}/x^d)\Phi(q^{-n}x,s)\theta(x)\,\mathrm{d}x.$$

Since the support of $\Phi(\cdot,s)$ is contained in a fixed compact set of F, there exists an integer n_0 such that the individual integrals $\eta_{a,n}(s)$ are 0 for $n\leqslant -n_0$. Using the change

of variables $x = 1/u$, one finds

$$\begin{aligned}\eta_{a,n}(s) &= \int_{q^{-1}<|x|<q} |x|^{s-1}\,\psi(aq^{dn}/x^d)\Phi(q^{-n}x,s)\theta(x)\,\mathrm{d}x \\ &= \int_{q^{-1}<|u|<q} |u|^{-s-1}\,\psi(aq^{nd}u^d)\Phi(q^{-n}/u,s)\theta(1/u)\,\mathrm{d}u.\end{aligned}$$

By Lemma 2.3.2, applied to the Schwartz function $u \mapsto |u|^{-s-1}\,\Phi(q^{-n}/u,s)\theta(1/u)$, there exists a real number c such that

$$|\eta_{a,n}(s)| \leqslant c(\left|aq^{nd}\right|)^{-1/d} \leqslant cq^{-n}\,|a|^{-1/d}\,.$$

(In fact, in the ultrametric case, there exists an integer n_1 such that these integrals are zero for $n \geqslant n_1$.) It follows that the series defining $\eta_a(s)$ is bounded term by term by

$$c\sum_{n\geqslant -n_0} q^{-n(s+1)}\,|a|^{-1/d} = c\,|a|^{-1/d}\,q^{n_0(s+1)}\frac{1}{1-q^{-(s+1)}}.$$

Consequently, the series $\eta_a(s) = \sum \eta_{a,n}(s)$ converges normally for $\mathrm{Re}(s+1) > 0$, locally uniformly in $a \in F^*$ and locally uniformly in s, proving the holomorphy in s. The lemma is thus proved. □

3. Compactifications of additive groups

In this chapter, we use methods of harmonic analysis to derive asymptotic formulas for the number of integral points of bounded height on partial (equivariant) compactifications of additive groups.

3.1. Setup and notation

3.1.1. Algebraic geometry. — Let G be the group scheme $\mathbf{G}_a^n$, and let X be a smooth projective equivariant compactification of G over a number field F. The geometry of such compactifications has been investigated in [7]. We recall the key facts. The boundary $X \setminus G$ decomposes as a union of F-irreducible divisors : $X \setminus G = \bigcup_{\alpha\in\mathscr{A}} D_\alpha$, forming a basis of the group $\mathrm{Pic}(X)$ of equivalence classes of divisors, and a basis of the monoid $\Lambda_{\mathrm{eff}}(X)$ of classes of effective divisors in $\mathrm{Pic}(X)$.

Since $\mathrm{Pic}(G) = 0$, line bundles on X have a G-linearization, which is unique up to a scalar since G carries no non-constant invertible functions. Thus, any line bundle possesses a meromorphic global section, unique modulo scalars, whose divisor does not meet G, hence is a linear combination of the D_α. Therefore, we will freely identify line bundles on X with divisors contained in the boundary, and with their classes in the Picard group.

Note that we do not assume that the divisors D_α are geometrically irreducible. Let $\bar{\mathscr{A}}$ be the set of irreducible components of $(X \setminus G)_{\bar F}$; this is a finite set with an action of the Galois group $\Gamma_F = \mathrm{Gal}(\bar F/F)$ whose set of orbits identifies with $\mathscr{A}$. More generally, for any extension E of F together with an embedding of $\bar F$ in $\bar E$, the set of orbits of $\bar{\mathscr{A}}$

under the natural action of $\mathrm{Gal}(\bar{E}/E)$ is identified with the set of irreducible components of $(X \setminus G)_E$. As above, the classes of these irreducible components form a basis of the Picard group $\mathrm{Pic}(X_E)$, as well as a basis of its effective cone $\Lambda_{\mathrm{eff}}(X_E)$.

For each $\alpha \in \mathscr{A}$, let F_α be the algebraic closure of F in the function field of D_α. It is a finite extension of F. After choosing a particular geometrically irreducible component of $D_{\alpha,\bar{F}}$ (*i.e.*, a specific element in the orbit in $\bar{\mathscr{A}}$ corresponding to α), we may view F_α as a subfield of $\bar{F}$; we write Γ_α for the Galois group $\mathrm{Gal}(\bar{F}/F_\alpha)$. The representation of the Galois group Γ_F on $\mathrm{Pic}(X_{\bar{F}})$ is the direct sum of the permutation modules $\mathrm{Ind}_{F_\alpha}^F[\mathbf{1}]$ obtained by inducing the trivial representation from $\mathrm{Gal}(\bar{F}/F_\alpha)$ to $\mathrm{Gal}(\bar{F}/F)$.

Let K_X be the canonical class of X, *i.e.*, the class of the divisor of any meromorphic differential form of top degree. In fact, up to multiplication by a scalar, there is a unique G-invariant meromorphic differential form ω_X on X; its restriction to G is proportional to the form $\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$. The anticanonical class K_X^{-1} is effective; indeed, writing $\sum_{\alpha\in\mathscr{A}} \rho_\alpha D_\alpha$ for *minus* the divisor of ω_X, we have $\rho_\alpha \geqslant 0$ for any α; in fact, $\rho_\alpha \geqslant 2$ ([7], Theorem 2.7).

We also recall that $\mathrm{H}^i(X, \mathscr{O}_X) = 0$ for $i > 0$. Indeed, X is birational to the projective space $\mathbf{P}^n$ and these cohomology groups are birational invariants of smooth proper varieties in characteristic zero.

3.1.2. Adelic metrics and heights. — Endow each line bundle $\mathscr{O}(D_\alpha)$ with a smooth adelic metric, as in 2.2.3 of [4]. The line bundles $\mathscr{O}(D_\alpha)$ have a canonical section which we denote by f_α.

For $\mathbf{s} \in \mathbf{C}^{\mathscr{A}} \simeq \mathrm{Pic}(X) \otimes \mathbf{C}$ and $\mathbf{x} = (\mathbf{x}_v)_v \in G(\mathbb{A}_F)$, we let

$$H(\mathbf{x}; \mathbf{s}) = \prod_{\alpha\in\mathscr{A}} \prod_{v} \|\mathsf{f}_\alpha\|_v (\mathbf{x}_v)^{-s_\alpha}.$$

When $\mathbf{s}$ corresponds to a very ample class λ in $\mathrm{Pic}(X)$, the restriction of $H(\cdot; \mathbf{s})$ to $G(F)$ is the standard (exponential) height relative to the projective embedding of X defined by λ. In particular, Northcott's theorem asserts that for any real number B, the set of $x \in G(F)$ such that $H(\mathbf{x}; \mathbf{s}) \leqslant B$ is finite. When all components s_α of $\mathbf{s}$ are positive, the corresponding line bundle λ belongs to the interior of the effective cone; by Prop. 4.3 of [3], this finiteness property still holds.

3.1.3. Partial compactifications. — A partial compactification of G is a smooth quasi-projective scheme U, containing G as an open subset, endowed with an action of G which extends the translation action on G. We will always assume, as we may, that U is the complement to a reduced divisor D in a smooth projective equivariant compactification X of G as above. The divisor D will be called the boundary divisor of U. We let also $\mathscr{A}_D$ to be the subset of $\mathscr{A}$ such that

$$D = \sum_{\alpha\in\mathscr{A}_D} D_\alpha.$$

The log-canonical class of U in $\mathrm{Pic}(X)$ is the class of $K_X + D$, the log-anticanonical class is its opposite. Since $\rho_\alpha \geqslant 2$ for all $\alpha \in \mathscr{A}$, $-(K_X + D)$ belongs to the interior of the effective cone of X, so is big.

We have introduced in [4], Definition 2.4.4, a virtual Γ_F-module $\mathrm{EP}(U)$. It is the difference of the Galois modules $\mathrm{H}^0(U_{\bar F}, \mathscr{O}^*)/\bar F^*$ and $\mathrm{Pic}(U)/\text{torsion}$ (both abelian groups are free of finite rank).

Lemma 3.1.4. — *The virtual representation* $\mathrm{EP}(U)$ *is given by*

$$-\sum_{\alpha\in\mathscr{A}\setminus\mathscr{A}_D} \mathrm{Ind}_{F_\alpha}^F[\mathbf{1}].$$

where $[\mathbf{1}]$ *is the abelian group* $\mathbf{Z}$ *together with the trivial action of* Γ_F*, and* $\mathrm{Ind}_{F_\alpha}^F$ *denotes the induction functor to* Γ_F *from its subgroup of finite index* Γ_α*.*

Proof. — One has $\mathrm{H}^0(U_{\bar F}, \mathbf{G}_m) = F^*$, because U contains G over which the result is already true. Moreover, the classes of the divisors $D_\alpha \cap U$, for $\alpha \notin \mathscr{A}_D$, form a basis of $\mathrm{Pic}(U_{\bar F})$; see Proposition 1.1 of [3] when $D = \varnothing$, but the proof holds for any equivariant embedding of G. □

3.1.5. Clemens complexes. — We have introduced in [4] various simplicial complexes to encode the combinatorial properties of the irreducible components of the boundary divisors D_α and their intersections. Let (V, Z) be a pair consisting of a smooth variety over a field F and of a divisor Z such that $Z_{\bar F}$ has strict normal crossings, *i.e.*, is a sum of smooth irreducible components which meet transversally.

Let $\bar{\mathscr{A}}$ be the set of irreducible components of $Z_{\bar F}$, together with its action of Γ_F; for $\alpha \in \bar{\mathscr{A}}$, let Z_α be the corresponding component. For any subset A of $\bar{\mathscr{A}}$, let

$$Z_A = \bigcap_{\alpha\in\mathscr{A}} Z_\alpha, \quad Z_A^\circ = Z_A \setminus \left(\bigcup_{\beta\notin A} Z_\beta\right).$$

The sets Z_A are closed in $V_{\bar F}$, the sets Z_A° form a partition of V in locally closed subsets. In particular, $Z_\varnothing^\circ = (V \setminus Z)_{\bar F}$. Unless they are empty, the sets Z_A and Z_A° are defined over F if and only if A is globally invariant under Γ_F.

The geometric Clemens complex $\mathscr{C}_{\bar F}(Z)$ of the pair (V, Z) has for vertices the elements of $\bar{\mathscr{A}}$; more generally, its faces are the irreducible components of the closed subsets Z_A, for all non-empty subsets $A \subset \bar{\mathscr{A}}$. An n-dimensional face corresponds to a component of codimension $n+1$ in V.

The geometric Clemens complex carries a natural simplicial action of the group Γ_F. The rational Clemens complex, $\mathscr{C}_F(Z)$, of the pair (V, Z) has for faces the Γ_F-invariant faces of $\mathscr{C}_{\bar F}(Z)$. A similar complex $\mathscr{C}_E(Z)$ can be defined for any extension E of F; we will apply this when $E = F_v$ is the completion of F at a place v of F.

For any place v of F, the v-analytic Clemens complex $\mathscr{C}_{F_v}^{\mathrm{an}}(Z)$ is then defined as the subcomplex of $\mathscr{C}_{F_v}(Z)$ whose faces correspond to irreducible components of intersections of irreducible components of Z_{F_v}, which contain F_v-rational points.

We will apply these considerations when $V = X$ and $Z = D = X \setminus U$, where X is a smooth projective equivariant compactification of G, and $U \subset X$ is a partial compactification. We require throughout this paper that over $\bar{F}$ the divisor $X \setminus G$ has strict normal crossings.

3.1.6. Notation from algebraic number theory. — Let $\mathrm{Val}(F)$ be the set of places of F. For $v \in \mathrm{Val}(F)$, let F_v be the completion of F at the place v, and $\mathbb{A}_F$ the adele ring of F. We also fix a finite set $S \subset \mathrm{Val}(F)$ containing the archimedean places. We write $\zeta_{F,v}$ for the local factor of Dedekind's zeta function at a place $v \in \mathrm{Val}(F)$, and ζ_F^S for the Euler product over places $v \notin S$. This product converges for $\mathrm{Re}(s) > 1$ and has a meromorphic continuation to the whole complex plane, with a single pole at $s = 1$; its residue at $s = 1$ is denoted by $\zeta_F^{S,*}(1)$.

3.1.7. Measures. — Let us fix a gauge form $\mathrm{d}\mathbf{x}$ on G, defined over F. For any place v of F, its absolute value is a Haar measure on $G(F_v)$, still denoted $\mathrm{d}\mathbf{x}$ (or $\mathrm{d}\mathbf{x}_v$). The product of these Haar measures is a Haar measure on $G(\mathbb{A}_F)$. Moreover, $G(F)$ is a discrete cocompact subgroup of covolume 1 (see [10] for the case $G = \mathbf{G}_a$; the general case follows from it).

The fixed adelic metrization of the line bundles $\mathscr{O}(D_\alpha)$ induces an adelic metrization on the canonical and log-canonical line bundles. As in Section 2.1.9 of [4], this gives rise, for any place v of F, to measures $\tau_{X,v}$ on the F_v-analytic manifold $X(F_v)$ and its restriction $\tau_{U,v}$ to the open submanifold $U(F_v)$ of $X(F_v)$. We also introduced the measure

$$\tau_{(X,D),v} = \|\mathsf{f}_D\|_v^{-1} \tau_{U,v}$$

on $U(F_v)$, where f_D is the canonical section of $\mathscr{O}_X(D)$.

By Theorem 2.4.7 of [4], the product of the local measures $\mathrm{L}_v(1, \mathrm{EP}(U))\tau_{U,v}$, for $v \notin S$, converges to a measure on the adelic space $U(\mathbb{A}_F^S)$ outside S. We then define

$$\tau_U^S = \mathrm{L}_*(1, \mathrm{EP}(U))^{-1} \prod_{v \notin S} \mathrm{L}_v(1, \mathrm{EP}(U))\tau_{U,v}.$$

Let v be a place in S and fix a decomposition group $\Gamma_v \subset \Gamma_F$ at v.

Let $\mathscr{A}_v = \bar{\mathscr{A}}/\Gamma_v$ and $\mathscr{A}_{D,v}$ be the set of orbits of the group Γ_v acting on $\bar{\mathscr{A}}$ and $\bar{\mathscr{A}}_D$. They are equal to the sets of F_v-irreducible components of $(X \setminus G)_{F_v}$ and of D_{F_v}, respectively. For each $\alpha \in \mathscr{A}$, the divisor $(D_\alpha)_{F_v}$ decomposes as a sum of irreducible components $D_{\alpha,w}$ indexed by the direct factors $F_{\alpha,w}$ of the algebra $F_\alpha \otimes_F F_v$. The canonical section f_α of $\mathscr{O}(D_\alpha)$ decomposes accordingly as a product $\prod \mathsf{f}_{(\alpha,w)}$. We shall identify $\mathscr{A}_v$ with the set of such pairs (α, w), and $\mathscr{A}_{D,v}$ with those such that $\alpha \in \mathscr{A}_D$.

Let A be a face of the analytic Clemens complex $\mathscr{C}_{F_v}^{\mathrm{an}}(X, D)$, that is a subset of $\mathscr{A}_{D,v}$ such that the intersection D_A of the divisors $D_{\alpha,w}$, for $(\alpha, w) \in A$, has a common F_v-rational point. Following the constructions in Sections 2.1.13–2.1.15 of [4], we have "residue measures" on the submanifold $D_A(F_v)$ of $X(F_v)$. As in Section 4.1 of that paper, we normalize these by incorporating the product $\prod_{\alpha \in A} \mathrm{c}_{F_\alpha,w}$ associated to the local fields $F_{\alpha,w}$, for $(\alpha, w) \in A$. The resulting measures are denoted by $\tau_{D_A,v}$.

3.1.8. Integral points. — Let S be a finite set of places of F containing the archimedean places. We are interested in "S-integral points of U". The precise definition depends on the choice of a model of U over the ring $\mathfrak{o}_{F,S}$ of S-integers in F; namely a quasi-projective scheme over $\operatorname{Spec}(\mathfrak{o}_{F,S})$ whose restriction to the generic fiber is identified with U. Given such a model, the S-integral points of U are the elements of $\mathscr{U}(\mathfrak{o}_{F,S})$, in other words those rational points of $U(F)$ which extend to a section of the structure morphism from $\mathscr{U}$ to $\operatorname{Spec}(\mathfrak{o}_{F,S})$. Fix such a model $\mathscr{U}$.

For any finite place v of F such that $v \notin S$, let $\mathfrak{u}_v = \mathscr{U}(\mathfrak{o}_{F,v})$ and let δ_v be the characteristic function of the subset $\mathfrak{u}_v \subset X(F_v)$. Consequently, *a point $x \in X(F)$ is an S-integral point of U if and only if one has $x \in \mathfrak{u}_v$ for any place v of F such that $v \notin S$*, equivalently, if and only if $\prod_{v\notin S} \delta_v(x) = 1$. We put $\delta_v \equiv 1$ when $v \in S$.

By the definition of an adelic metric, there exists a finite set of places T, a flat projective model $\mathscr{X}$ over $\mathfrak{o}_{F,T}$ satisfying the following properties:

– $\mathscr{X}$ is a smooth equivariant compactification of the $\mathfrak{o}_{F,T}$-group scheme G;
– for any $\alpha \in \mathscr{A}$, the closure $\mathscr{D}_\alpha$ of D_α is a divisor on $\mathscr{X}$;
– the boundary $\mathscr{X} \setminus G$ is the union of these divisors $\mathscr{D}_\alpha$;
– for any α, the section f_α extends to a global section of the line bundle $\mathscr{O}(\mathscr{D}_\alpha)$ on $\mathscr{X}$ whose divisor is precisely $\mathscr{D}_\alpha$.

Since any isomorphism between two $\mathfrak{o}_{F,S}$-schemes of finite presentation extends uniquely to an isomorphism over an open subset of $\operatorname{Spec} \mathfrak{o}_{F,S}$, we may assume that after restriction to $\operatorname{Spec}(\mathfrak{o}_{F,T})$, $\mathscr{U}$ is the complement in $\mathscr{X}$ to the Zariski closure $\mathscr{D} = \sum_{\alpha\in\mathscr{A}_D} \mathscr{D}_\alpha$ of D in $\mathscr{X}$. For all places $v \notin T \cup S$, one thus has $\mathfrak{u}_v = \mathscr{U}(\mathfrak{o}_v)$.

3.1.9. Height zeta function. — We proceed to study of the distribution of S-integral points of U with respect to heights. In fact, we only study those integral points which belong to $G(F)$; moreover, we consider the heights with respect to all line bundles in the Picard group at the same time. The *height zeta function* is defined as

$$\mathrm{Z}(\mathbf{s}) = \sum_{\mathbf{x}\in G(F)\cap\mathscr{U}(\mathfrak{o}_{F,S})} H(\mathbf{x};\mathbf{s})^{-1},$$

whenever it converges. It follows from Proposition 4.5 in [3] that there exists a non-empty open subset $\Omega \subset \operatorname{Pic}(X)_{\mathbf{R}}$ such that $\mathrm{Z}(\mathbf{s})$ converges absolutely to a bounded holomorphic function in the tube domain $\mathsf{T}(\Omega) = \Omega + \mathrm{i}\operatorname{Pic}(X)_{\mathbf{R}}$.

Using the functions δ_v defined above, it follows that

$$\mathrm{Z}(\mathbf{s}) = \sum_{\mathbf{x}\in G(F)} \prod_{v\in\mathrm{Val}(F)\setminus S} \big(\delta_v(\mathbf{x}_v) \prod_{\alpha\in\mathscr{A}} \|\mathsf{f}_\alpha(\mathbf{x}_v)\|_v^{s_\alpha}\big) \times \prod_{v\in S} \big(\prod_{\alpha\in\mathscr{A}} \|\mathsf{f}_\alpha(\mathbf{x}_v)\|_v^{s_\alpha}\big).$$

3.1.10. Fourier transforms. — We write $\langle\cdot,\cdot\rangle$ for the bilinear pairing on G

$$\langle(x_1,\ldots,x_n),(y_1,\ldots,y_n)\rangle = \sum x_i y_i.$$

Then, the maps

$$G(F_v)\times G(F_v) \to \mathbf{C}^*, \quad (\mathbf{a},\mathbf{x}) \mapsto \psi_v(\langle\mathbf{a},\mathbf{x}\rangle)$$

and

$$G(\mathbb{A}_F) \times G(\mathbb{A}_F) \to \mathbf{C}^*, \quad (\mathbf{a}, \mathbf{x}) \mapsto \psi(\langle \mathbf{a}, \mathbf{x} \rangle)$$

are perfect pairings.

For each place v of F and $\mathbf{a} \in G(F_v)$, define

$$\hat{H}_v(\mathbf{a}; \mathbf{s}) = \begin{cases} \displaystyle\int_{G(F_v)} \delta_v(\mathbf{x}) \prod_{\alpha \in \mathscr{A}} \|\mathsf{f}_\alpha(\mathbf{x})\|_v^{s_\alpha} \, \psi_v(\langle \mathbf{a}, \mathbf{x} \rangle) \, \mathrm{d}\mathbf{x} & \text{if } v \notin S \\ \displaystyle\int_{G(F_v)} \prod_{\alpha \in \mathscr{A}} \|\mathsf{f}_\alpha(\mathbf{x})\|_v^{s_\alpha} \, \psi_v(\langle \mathbf{a}, \mathbf{x} \rangle) \, \mathrm{d}\mathbf{x} & \text{otherwise,} \end{cases}$$

where the integrals are taken with respect to the chosen Haar measure on $G(F_v)$. Moreover, for any $\mathbf{a} = (\mathbf{a}_v)_v \in G(\mathbb{A}_F)$, we define

$$\hat{H}(\mathbf{a}; \mathbf{s}) = \prod_{v \in \mathrm{Val}(F)} \hat{H}_v(\mathbf{a}_v; \mathbf{s}).$$

Formally, we can write down the Poisson summation formula for the locally compact group $G(\mathbb{A}_F)$ and its discrete cocompact subgroup $G(F)$. Since $G(F)$ has covolume 1,

$$\mathrm{Z}(\mathbf{s}) = \sum_{\mathbf{a} \in G(F)} \hat{H}(\mathbf{a}; \mathbf{s}). \tag{3.1.11}$$

Below, we will establish a series of analytic estimates guaranteeing that the Poisson summation formula can be applied and we will show that its right hand side provides a meromorphic continuation of the height zeta function.

3.2. Fourier transforms (trivial character)

In our paper [4], we have established the analytic properties of the Fourier transform at the trivial character, *i.e.*, of the local integrals

$$\hat{H}_v(0; \mathbf{s}) = \int_{G(F_v)} \delta_v(\mathbf{x}) \prod_{\alpha \in \mathscr{A}} \|\mathsf{f}_\alpha(\mathbf{x})\|_v^{s_\alpha} \, \mathrm{d}\mathbf{x}$$

for $v \in \mathrm{Val}(F)$, and of their "Euler product"

$$\hat{H}(0; \mathbf{s}) = \prod_{v \in \mathrm{Val}(F)} \hat{H}_v(0; \mathbf{s}).$$

We now summarize these results.

If a and b are real numbers, we define $\mathsf{T}_{>a}$, *resp.* $\mathsf{T}_{(a,b)}$, as the set of $s \in \mathbf{C}$ such that $a < \mathrm{Re}(s)$, *resp.* $a < \mathrm{Re}(s) < b$. We write $\mathsf{T}_{>a}^{\mathscr{A}}$ for the set of families of elements of $\mathsf{T}_{>a}$ indexed by the set $\mathscr{A}$, etc.

3.2.1. Absolute convergence of the local integrals. — We begin by stating the domain of absolute convergence of the local integrals $\hat{H}_v(0;\mathbf{s})$, as well as their meromorphic continuation. We recall that $\rho=(\rho_\alpha)_{\alpha\in\mathscr{A}}$ is the vector of positive integers such that $\sum_{\alpha\in\mathscr{A}}\rho_\alpha D_\alpha$ is an anticanonical divisor.

Lemma 3.2.2. — *Let v be a place of F. The integral $\hat{H}_v(0;(s_\alpha+\rho_\alpha-1)_{\alpha\in\mathscr{A}})$ converges for $s\in\mathsf{T}^{\mathscr{A}}_{>0}$ and defines a holomorphic function on $\mathsf{T}^{\mathscr{A}}_{>0}$. The holomorphic function*

$$\mathbf{s}\mapsto\prod_{(\alpha,w)\in\mathscr{A}_v}\zeta_{F_\alpha,v}(s_\alpha)^{-1}\hat{H}_v(0;(s_\alpha+\rho_\alpha-1))$$

extends to a holomorphic function on $\mathsf{T}^{\mathscr{A}}_{>-1}$.

Proof. — With the notation of Lemma 4.1.1 in [4], we have

$$\mathrm{d}\mathbf{x}=\mathrm{d}\tau_{(X,D),v}=\prod_{\alpha\in\mathscr{A}}\|\mathsf{f}_\alpha(\mathbf{x})\|_v^{-\rho_\alpha}\,\mathrm{d}\tau_X.$$

Consequently, the convergence assertion, follows from this Lemma by taking for Φ the function δ_v introduced above. The meromorphic continuation is then a particular case of Proposition 4.1.2 of that paper. □

3.2.3. Places in S. — We fix a place $v\in\mathrm{Val}(F)$ such that $v\in S$. For $\lambda\in\mathbf{R}^{\mathscr{A}}_{>0}$, let

$$a(\lambda,\rho)=\max_{\alpha\in\mathscr{A}}\frac{\rho_\alpha-1}{\lambda_\alpha}$$

and let $\mathscr{A}(\lambda,\rho)$ be the set of all $\alpha\in\mathscr{A}$ where the maximum is achieved. By Lemma 3.2.2, the integral defining $\hat{H}_v(0;s\lambda)$ converges for any complex number s such that $\mathrm{Re}(s)>a(\lambda,\rho)$, defines a holomorphic function of s in the tube domain $\mathsf{T}_{>a(\lambda,\rho)}$, and has a meromorphic continuation to a tube $\mathsf{T}_{>a(\lambda,\rho)-\delta}$, for some positive real number $\delta>1/\max(\lambda_\alpha)$, with a pole of order at most $\#\mathscr{A}_v$ at $s=a(\lambda,\rho)$.

In order to state a precise answer, let us introduce the simplicial complex $\mathscr{C}^{\mathrm{an}}_{F_v,(\lambda,\rho)}(X\setminus G)$ obtained from $\mathscr{C}^{\mathrm{an}}_{F_v}(X\setminus G)$ by removing all faces containing a vertex $(\alpha,w)\in\mathscr{A}_v$ such that $\rho_\alpha-1<a(\lambda,\rho)\lambda_\alpha$.

Proposition 3.2.4. — *There exist a positive real number δ, and for each face A of $\mathscr{C}^{\mathrm{an}}_{F_v,(\lambda,\rho)}(X\setminus G)$ of maximal dimension, a holomorphic function φ_A defined on $\mathsf{T}_{>a(\lambda,\rho)-\delta}$ with polynomial growth in vertical strips such that*

$$\varphi_A(a(\lambda,\rho))=\int_{D_A(F_v)}\prod_{(\alpha,w)\notin A}\left\|\mathsf{f}_{(\alpha,w)}(\mathbf{x})\right\|_v^{a(\lambda,\rho)\lambda_\alpha-1}\mathrm{d}\tau_{D_A,v}(\mathbf{x})$$

and such that for any $s\in\mathsf{T}_{>a(\lambda,\rho)}$, one has

$$\hat{H}_v(0;s\lambda)=\sum_A\varphi_A(s)\prod_{(\alpha,w)\in A}\zeta_{F_{\alpha,w}}(\lambda_\alpha(s-a(\lambda,\rho))),$$

where the sum ranges over the faces A of $\mathscr{C}^{\mathrm{an}}_{F_v,(\lambda,\rho)}(D)$ *of maximal dimension. In particular, the order of the pole of* $\hat{H}_v(0;\lambda s)$ *at* $s=a(\lambda,\rho)$ *is given by*

$$b_v(\lambda,\rho)=1+\dim\mathscr{C}^{\mathrm{an}}_{F_v,(\lambda,\rho)}(D).$$

Proof. — This is a special case of [4], Proposition 4.1.4. □

3.2.5. Places outside S; Denef's formula. — When $v\notin S$, the statement of Lemma 3.2.2 does not take into account the compactly supported function δ_v on $U(F_v)$. The same proof shows that the inequalities $\mathrm{Re}(s_\alpha)>\rho_\alpha-1$, for $\alpha\in\mathscr{A}_D$, suffice to ensure the absolute convergence of the integral $\hat{H}_v(0;\mathbf{s})$.

Moreover, for places of good reduction, *i.e.*, places $v\notin T\cup S$, we may apply Denef's formula (Proposition 4.1.6 of [4]) and obtain the explicit formula

$$\hat{H}_v(0;(s_\alpha))=(q_v^{-1}\mu_v(\mathfrak{o}_v))^{\dim X}\sum_{A\subset(\bar{\mathscr{A}}\setminus\bar{\mathscr{A}}_D)/\Gamma_v}\#\mathscr{D}^0_A(k_v)\prod_{\alpha\in A}\frac{q_v^{f_\alpha}-1}{q_v^{f_\alpha(s_\alpha-\rho_\alpha)}-1}. \tag{3.2.6}$$

(An element $\alpha\in A$ corresponds to an F_v-irreducible component of $(X\setminus G)_{F_v}$ which is not contained in D_{F_v}. By the good reduction hypothesis, this component is split over an unramified extension of F_v, of degree f_α.)

3.2.7. Euler product. — We have explained in [4] how to derive from this explicit formula the analytic behavior of the infinite product of $\hat{H}_v(0,\mathbf{s})$ over all places $v\notin S$. Following the proof of Proposition 4.3.4 in that paper, we obtain:

Proposition 3.2.8. — *The infinite product*

$$\hat{H}^S(0;\mathbf{s})=\prod_{v\notin S}\hat{H}_v(0;\mathbf{s})$$

converges absolutely for any $\mathbf{s}\in\mathbf{C}^{\mathscr{A}}$ *such that* $\mathrm{Re}(s_\alpha)>\rho_\alpha+1$ *for all* $\alpha\in\mathscr{A}\setminus\mathscr{A}_D$. *Moreover, there exists a holomorphic function* φ^S *defined on the set of* $\mathbf{s}\in\mathbf{C}^{\mathscr{A}}$ *such that* $\mathrm{Re}(s_\alpha)>\rho_\alpha+1/2$ *for* $\alpha\in\mathscr{A}\setminus\mathscr{A}_D$ *which has polynomial growth in vertical strips and satisfies*

$$\hat{H}^S(0;\mathbf{s})=\varphi^S(\mathbf{s})\prod_{\alpha\notin\mathscr{A}_D}\zeta^S_{F_\alpha}(s_\alpha-\rho_\alpha).$$

Moreover, for any point $\mathbf{s}\in\mathbf{C}^{\mathscr{A}}$ *such that* $s_\alpha=\rho_\alpha+1$ *for all* $\alpha\notin\mathscr{A}_D$, *one has*

$$\varphi^S(\mathbf{s})=\prod_{\alpha\notin\mathscr{A}_D}\zeta^{S,*}_{F_\alpha}(1)\int_{U(\mathbb{A}^S_F)}\prod_{v\notin S}\delta_v(\mathbf{x})\prod_{\alpha\notin\mathscr{A}_D}\prod_{v\notin S}\|\mathsf{f}_\alpha(\mathbf{x})\|_v^{s_\alpha-1}\,\mathrm{d}\tau_U(\mathbf{x}).$$

3.3. The Fourier transforms at non-trivial characters

3.3.1. Vanishing of the Fourier transforms outside of a lattice. — We proceed with an elementary, but important, remark — we assume that the written integrals converge absolutely.

LEMMA 3.3.2. — *For each finite place v, there exists a compact open subgroup $\mathfrak{d}_{X,v} \subset G(F_v)$ such that $\hat{H}_v(\mathbf{a};\mathbf{s}) = 0$ for $\mathbf{a} \notin \mathfrak{d}_{X,v}$. Moreover, $\mathfrak{d}_{X,v} = G(\mathfrak{o}_v)$ for almost all finite places v.*

There exists a lattice $\mathfrak{d}_X \subset G(F)$ such that $\hat{H}(\mathbf{a};\mathbf{s}) = 0$ for $\mathbf{a} \notin \mathfrak{d}_X$.

Proof. — For each finite place v, the function $\mathbf{x} \mapsto H_v(\mathbf{x};\mathbf{s})$ on $G(F_v)$ is invariant under the action of an open subgroup of $G(F_v)$, which equals $G(\mathfrak{o}_v)$ for almost all v. (See [3], Proposition 4.2.) Consequently, the Fourier transform vanishes at any character whose restriction to this open subgroup is non-trivial. This establishes the claim. □

3.3.3. Archimedean places: integration by parts. — In order to be able to prove the convergence of the right hand side of the Poisson formula (Eq. (3.1.11)) we need to improve the decay at infinity of the Fourier transforms at archimedean places of F.

Let v be such a place. As in [3], we use integration by parts with respect to vector fields on X which extend the invariant vector fields on G. According to Prop. 2.1 in *loc. cit.*, any invariant vector field on G extends uniquely to a regular vector field ∂^X on X; moreover, Prop. 2.2 there asserts that for any local equation z_α of a boundary component D_α, $z_\alpha^{-1}\partial^X z_\alpha$ is regular along D_α. As in Prop. 8.4 of this paper, we can perform repeated integration by parts and write, for $\mathbf{s}$ in the domain of absolute convergence:

$$\hat{H}_v(\mathbf{a};\mathbf{s}) = (1+\|\mathbf{a}\|_v)^{-N} \int_{G(F_v)} H_v(\mathbf{x};\mathbf{s})^{-1}\psi_v(\langle\mathbf{a};\mathbf{x}\rangle) h_N(\mathbf{a};\mathbf{x};\mathbf{s})\,\mathrm{d}\mathbf{x} \tag{3.3.4}$$

where h_N is a smooth function on $X(F_v)$ which admits a uniform upper-bound of the form

$$|h_N(\mathbf{a};\mathbf{x};\mathbf{s})| \ll (1+\|\mathbf{s}\|)^N.$$

The important observation to make is that these integration by parts preserve the form of the Fourier integrals when they are written in local coordinates. Consequently, we can apply the techniques developed in Section 3.4 to the integral expression (3.3.4) of the Fourier transform. This shows that the upper-bounds for the meromorphic continuation established in that Section can be improved by a factor $(1+\|s\|)^N/(1+\|a\|_v)^N$, where N is an arbitrary positive integer.

3.3.5. Places outside S. — Let $f_\mathbf{a}$ be the rational function on X corresponding to the linear form $\langle\mathbf{a},\cdot\rangle$ on G. Its divisor takes the form

$$\operatorname{div}(f_\mathbf{a}) = E(f_\mathbf{a}) - \sum_{\alpha\in\mathscr{A}} d_\alpha(\mathbf{a}) D_\alpha,$$

with $d_\alpha(\mathbf{a}) \geqslant 0$ for all $\alpha \in \mathscr{A}$, and $E(f_\mathbf{a})$ is the Zariski closure in X of the hyperplane with equation $\langle\mathbf{a},\cdot\rangle = 0$ in G. (See [3], Lemma 1.4.) For any $\mathbf{a}$, let $\mathscr{A}_0^D(\mathbf{a})$ be the set of $\alpha \in \mathscr{A} \setminus \mathscr{A}_D$ such that $d_\alpha(\mathbf{a}) = 0$. We also define $T(\mathbf{a})$ to be the union of S, T and of the set of finite places $v \notin S \cup T$ such that $\mathbf{a}$ reduces to 0 modulo v.

Proposition 3.3.6. — *There exists a constant $C(\varepsilon)$ independent of $\mathbf{a} \in \mathfrak{d}_X$ such that for any $v \notin T(\mathbf{a})$ and any $\mathbf{s} \in \mathbf{C}^{\mathscr{A}}$ such that $\mathrm{Re}(s_\alpha) > \rho_\alpha - \frac{1}{2} + \varepsilon$ for any α,*

$$\left| 1 - \hat{H}_v(\mathbf{a}; \mathbf{s}) \prod_{\alpha \in \mathscr{A}_0^D(\mathbf{a})} (1 - q_v^{-f_\alpha(1+s_\alpha-\rho_\alpha)}) \right| \leqslant C(\varepsilon) q_v^{-1-\varepsilon}.$$

Proof. — When $X = U$, *i.e.*, when U is projective and $D = \varnothing$, this has been proved in [3], see Proposition 10.2 and §11 there. A straightforward adaptation of that proof establishes the general case: by the definition of δ_v, we split the integral as a sum of integrals over the residue classes in $\mathscr{U}(k_v)$ and each of these integrals is computed in [3], leading to the asserted formula. □

As a consequence, we obtain the following meromorphic continuation of the infinite product over places $v \notin S$ of $\hat{H}_v(\mathbf{a}; \mathbf{s})$.

Corollary 3.3.7. — *For any $\varepsilon > 0$ and $\mathbf{a} \in \mathfrak{d}_X \setminus \{0\}$ there exists a holomorphic bounded function $\varphi(\mathbf{a}; \cdot)$ on $\mathsf{T}^{\mathscr{A}}_{>-1/2+\varepsilon}$ such that for any $\mathbf{s} \in \mathsf{T}^{\mathscr{A}}_{>0}$,*

$$\hat{H}^S(\mathbf{a}; \mathbf{s} + \rho) = \prod_{v \notin S} \hat{H}_v(\mathbf{a}; \mathbf{s} + \rho) = \varphi(\mathbf{a}; \mathbf{s}) \prod_{\alpha \in \mathscr{A}_0^D(\mathbf{a})} \zeta_{F_\alpha}(1 + s_\alpha).$$

Moreover, there exist a real number $C(\varepsilon)$ such that one has the uniform estimate

$$|\varphi(\mathbf{a}; \mathbf{s})| \leqslant C(\varepsilon)(1 + \|\mathbf{a}\|_\infty)^\varepsilon.$$

3.4. The Fourier transforms at non-trivial characters (places in S)

Here we study the Fourier transforms for a place $v \in S$, when the character $\mathbf{a}$ is non-trivial. We will prove uniform estimates in $\mathbf{a}$. When no confusion can arise, we will remove the index v from the notation. Written in local charts of the compactification, our integrals take the form

$$\int_{\mathscr{U}_A} \prod_{\alpha \in A} |x_\alpha|^{\lambda_\alpha s - \rho_\alpha} \psi_v(f_{\mathbf{a}}(\mathbf{x})) \theta_A(\mathbf{x}; \mathbf{s}) \prod_{\alpha \in A} \mathrm{d}x_\alpha \, \mathrm{d}\mathbf{y},$$

where $\mathbf{x} = ((x_\alpha)_{\alpha \in A}, \mathbf{y})$ is a system of local coordinates for $\mathbf{x}$ in a neighborhood of a point of $D_A^\circ(F_v)$, and θ_A a smooth function with compact support in $\mathbf{x}$, which is holomorphic and has polynomial growth in vertical strips with respect to the parameter $\mathbf{s}$. See Section 3.2 of [4] for more details. Note that the stratum A of the F_v-analytic Clemens complex of $X \setminus G$ consists of elements $\tilde{\alpha} = (\alpha, w) \in \mathscr{A}_v$, the local coordinate $x_{\tilde{\alpha}}$ is an element of the local field $F_{\alpha,w}$; these coordinates are completed by the family $\mathbf{y} = (y_\beta)$. In the following exposition, we assume throughout that all irreducible components of $X \setminus G$ are geometrically irreducible, the general case is a straightforward notational adaptation.

The analytic properties of this integral are determined by the divisor of the rational function $f_{\mathbf{a}}$. The analysis is simpler if this divisor has strict normal crossings; we can reduce to this case using embedded resolution of singularities. To obtain uniform estimates, we do this in families.

The functions $\mathbf{a} \mapsto d_\alpha(\mathbf{a})$ on $G \setminus \{0\}$ are constructible, upper semi-continuous (they do not increase under specialization), and descend to the projective space $\mathbf{P}^{n-1}$ of lines in G.

Lemma 3.4.1 (Application of resolution of singularities). — *There exists a decomposition $\mathbf{P}^{n-1} = \coprod_i P_i$ of $\mathbf{P}^{n-1}$ in locally closed subsets, and, for each i, a map $Y_i \to P_i \times X$ which is a composition of blowups whose centers lie over a nowhere dense subset of $P_i \times (U \setminus G)$ and are smooth over P_i, such that the divisor of the rational functions $f_\mathbf{a}$ on Y_i is a relative divisor with normal crossings on P_i.*

Proof. — First apply Hironaka's theorem and perform an embedded resolution of singularities of the pair $(X, |\mathrm{div}(f_\mathbf{a})|)$ over the function field of $\mathbf{P}^{n-1}$. Then spread out this composition of blow-ups with smooth centers by considering the composition of the blow-ups of their Zariski closures in $\mathbf{P}^{n-1} \times X$. Let P_1 be the largest open subset of $\mathbf{P}^{n-1}$ over which all of these centers are smooth; it is dense. We also observe that the map $Y_1 \to P_1 \times X$ is an isomorphism over G, as well as over the generic points of $X \setminus G$, because the divisor of $f_\mathbf{a}$ is smooth there.

We repeat this procedure for each irreducible component of $\mathbf{P}^{n-1} \setminus P_1$. This process ends by noetherian induction. □

Let us return to our integrals. We fix one of the strata, say P, and derive uniform estimates for $[\mathbf{a}]$ in P. Note that all coefficients $d_\alpha(\mathbf{a})$ are constant on P; they will be denoted d_α. We write $\pi\colon Y \to P \times X$ for the resolution of singularities introduced in the Lemma. Considering the normalization $\bar{P}$ of the closure of P in $\mathbf{P}^{n-1}$, and the successive blow-ups along the Zariski closure of the successive centers (which may not be smooth on $\bar{P}$ anymore), we extend it to a map $\bar{\pi}\colon \bar{Y} \to \bar{P} \times X$. We change variables and write our integral as an integral on the fiber $\bar{Y}_{[\mathbf{a}]}$ over the point $[\mathbf{a}] \in P$. This is relatively innocuous as long as $[\mathbf{a}]$ lies in a compact subset of $P(F_v)$; when $[\mathbf{a}]$ approaches the boundary $\partial P = \bar{P} \setminus P$, the upper-bounds we obtain increase, but at most polynomially in the distance to the boundary.

After resolution of singularities, we can write the divisor $\mathrm{div}(f_\mathbf{a})$ as

$$-\sum_{\alpha \in A} d_\alpha D_\alpha + \sum_{\beta \in B} e_\beta E_\beta,$$

where the divisors D_α are the strict transforms of the one with the same name, and the divisors E_β are either the closure in Y of the hyperplane $G \cap \mathrm{div}(f_\mathbf{a})$ of G (for $\beta = 0 \in B$), or exceptional divisors of the resolution of singularities (for all other values of β). The integers e_β are unknown a priori, but do not depend on $[\mathbf{a}] \in P$. Moreover, the divisor of the Jacobian of π has the form

$$\sum_{\beta \in B} \iota_\beta E_\beta,$$

where $\iota_\beta > 0$ unless $\beta = 0$. For each $\alpha \in A$, let $d_{\alpha,\beta}$ be nonnegative integers such that

$$\pi^* D_\alpha = D_\alpha + \sum_{\beta \in B} d_{\alpha,\beta} E_\beta.$$

For each $\beta \in B$ we set

$$\lambda_\beta(s) = \sum_{\alpha\in A} d_{\alpha,\beta}(\lambda_\alpha s - \rho_\alpha) + \iota_\beta.$$

We observe that $\lambda_0 \equiv 0$, but that λ_β is positive at $\sigma = \max(\rho_\alpha/\lambda_\alpha)$, otherwise.

Let us consider local coordinates $((x_\alpha)_{\alpha\in A}, (y_\beta)_{\beta\in B}, (z_\gamma)_{\gamma\in C})$ around $\mathbf{x}$, where for each α, x_α is a local equation of D_α, for each $\beta \in B$, y_β is a local equation of E_β. By construction, the function

$$u_{\mathbf{a}} \colon x \mapsto \prod_{\alpha\in A} x_\alpha^{d_\alpha} \prod_{\beta\in B} y_\beta^{-e_\beta} f_{\mathbf{a}}$$

is regular in codimension 1, hence regular, on $\bar{P} \times \mathscr{U}_A$, and it does not vanish on $P \times \mathscr{U}_A$. Moreover, $u_{\mathbf{a}}$ is homogeneous of degree 1 in $\mathbf{a}$. Consequently, and maybe up to shrinking $\mathscr{U}_A$ a little bit, there exists a real number κ such that $u_{\mathbf{a}}$ admits uniform lower- and upper-bounds of the form

$$\|\mathbf{a}\| \, d([\mathbf{a}], \partial P)^\kappa \ll |u_{\mathbf{a}}(\mathbf{x})| \ll \|\mathbf{a}\| \,,$$

for $\mathbf{x} \in \mathscr{U}_A$ and $\mathbf{a} \in P$.

Our integrals now take the form

$$\int_{\mathscr{U}_A} \prod_{\alpha\in A} |x_\alpha|^{\lambda_\alpha s - \rho_\alpha} \prod_{\beta\in B} |y_\beta|^{\lambda_\beta(s)} \, \psi_v(u_{\mathbf{a}} \prod x_\alpha^{-d_\alpha} \prod y_\beta^{e_\beta}) \theta_A(\mathbf{x}; \mathbf{s}) \prod_{\alpha\in A} \mathrm{d}x_\alpha \prod_{\beta\in B} \mathrm{d}y_\beta \prod_{\gamma\in C} \mathrm{d}z_\gamma.$$

If some of the d_α are zero, these variables do not appear in the argument of the character ψ. We first integrate with respect to these variables, applying the method used for the trivial character to obtain a meromorphic continuation with poles at most as in the product $\prod_{d_\alpha=0} \zeta_{F_\alpha}(\lambda_\alpha s - \rho_\alpha + 1)$. Let

$$\sigma = \max_{d_\alpha=0} (\rho_\alpha - 1)/\lambda_\alpha$$

be the abscissa of the largest pole of this product.

We now explain why having some positive d_α implies *holomorphic* continuation with respect to the corresponding variables s_α. The discussion distinguishes two cases.

a) *All e_β are* $\leqslant 0$. We first integrate with respect to a variable x_α such that $d_\alpha > 0$ and use Lemma 2.4.1. For all α', the upper-bound provided by this Lemma increases the exponent of $x_{\alpha'}$ by $d_{\alpha'}/d_\alpha$.

b) *Some e_β is positive.* We first integrate with respect to these variable y_β such that $e_\beta > 0$. From the bound for oscillatory integrals that we established in Proposition 2.3.6, we obtain an upper-bound for the inner integral of the form

$$\|\mathbf{a}\|^{-\kappa} \prod_\alpha |x_\alpha|^{\kappa d_\alpha} \,,$$

where κ is a positive real number. There is a gain κd_α in the exponent of $|x_\alpha|$.

In both cases, there exists a positive real number δ such that the remaining integral converges absolutely, and uniformly, to a holomorphic function on the half-plane $\mathrm{Re}(s) > \sigma - \delta$, with polynomial decay in terms of $\|\mathbf{a}\|$. As a consequence, we obtain:

PROPOSITION 3.4.2. — *Let* $\mathbf{a}$ *be a non-trivial character and* $v \in S$. *Then there exist holomorphic functions* $\varphi_{A,v}$, *defined for* $\mathrm{Re}(s) > \sigma - \delta$, *such that*

$$\hat{H}_v(\mathbf{a}; s\lambda) = \sum_{\substack{A \in \mathscr{C}^{\mathrm{an}}_{F_v}(X \setminus G) \\ d_\alpha(\mathbf{a}) = 0\, \forall \alpha \in A \\ A \text{ maximal}}} \varphi_{A,v}(\mathbf{a}; s) \prod_{\tilde{\alpha} = (\alpha, w) \in A} \zeta_{F_{\tilde{\alpha}}}(\lambda_\alpha s - \rho_\alpha + 1).$$

Moreover, $\varphi_{A,v}$ *satisfy upper-bounds of the form*

$$|\varphi_{A,v}(\mathbf{a}; s)| \ll \frac{(1 + |s|)^\kappa}{d_v([\mathbf{a}], \partial P_{\mathbf{a}})^\kappa \, \|\mathbf{a}\|_v^{\kappa'}},$$

where $P_{\mathbf{a}}$ *is the stratum of* $\mathbf{P}^{n-1}$ *containing the line* $[\mathbf{a}]$, $d_v(([\mathbf{a}], \partial P_{\mathbf{a}})$ *the* v*-adic distance of* $[\mathbf{a}]$ *to the boundary* $\partial P_{\mathbf{a}} = \overline{P_{\mathbf{a}}} \setminus P_{\mathbf{a}}$, *and* κ, κ' *positive absolute constants.*

Note that in this statement, the subsets A over which the decomposition of $\hat{H}_v$ runs are the maximal faces of the sub-complex of the analytic Clemens complex (depending on $\mathbf{a}$) given by the vanishing of the coefficients $d_\alpha(\mathbf{a})$. These faces need not all have the same dimension.

3.5. Application of the Poisson summation formula

3.5.1. Meromorphic continuation of the height zeta function. — From now on, we consider the case where the height is attached to the log-anticanonical line bundle $-(K_X + D)$ with $D = X \setminus U$. Let $\lambda = \rho - \sum_{\alpha \in \mathscr{A}_D} D_\alpha$ be the corresponding class. In that case, the abscissa of convergence of the height zeta function is $\sigma = 1$.

Recall that we have decomposed the projective space of non-trivial characters $\mathbf{a}$ (modulo scalars) as a disjoint union of locally closed strata (P_i) and that on each stratum we obtained a uniform meromorphic continuation of $\hat{H}(\mathbf{a}; \mathbf{s})$. The height zeta function can then be expressed as

$$\mathrm{Z}(s) = \mathrm{Z}_0(s) + \sum_P \mathrm{Z}_P(s)$$

where $\mathrm{Z}_0(s) = \hat{H}(0; s\lambda)$ is the term of the right hand side of the Poisson formula (3.1.11) corresponding to the trivial character, the summation is over all strata, and the contribution from stratum P is

$$\mathrm{Z}_P(s) = \sum_{\substack{\mathbf{a} \neq 0 \\ [\mathbf{a}] \in P}} \hat{H}(\mathbf{a}; s\lambda).$$

The following lemma implies that each of Z_P converges absolutely for $\mathrm{Re}(s) > 1$ and admits a meromorphic continuation to $\mathrm{Re}(s) > 1 - \delta$, for some $\delta > 0$.

LEMMA 3.5.2. — *Let* Z *be a closed subvariety of a projective space* $\mathbf{P}^{n-1}$ *over the number field* F. *For each place* $v \in S$, *let* $d_v(\cdot, Z)$ *denote the* v*-adic distance of a point in* $\mathbf{P}^{n-1}(F_v)$ *to the subset* $Z(F_v)$.

Let $\mathfrak{d}$ be an $\mathfrak{o}_F$-lattice in F^n and let $\|\cdot\|_\infty$ be any norm on the real vector space $F^n \otimes_{\mathbf{Q}} \mathbf{R}$. Then, there are positive constants C and κ such that

$$\prod_{v\in S} d_v([\mathbf{a}], Z) \geqslant C(1 + \|\mathbf{a}\|_\infty)^{-\kappa}$$

for any $\mathbf{a} \in \mathfrak{d} \setminus \{0\}$ such that $[\mathbf{a}] \notin Z(F)$.

Proof. — Let Φ be a family of homogeneous polynomials defining Z set-theoretically in $\mathbf{P}^{n-1}$. Let $\tilde{Z}$ be the cone over Z in $\mathbf{A}^n$ defined by the same polynomials. We may assume that they have coefficients in $\mathfrak{o}_F$ and that all their degrees are equal to an integer d. For any $v \in S$, the distance of a point $[\mathbf{a}] \in \mathbf{P}^{n-1}(F)$ to $Z(F_v)$ satisfies

$$\log d_v([\mathbf{a}], Z) \approx \max_{\varphi\in\Phi} \log \|\varphi\|_v ([\mathbf{a}]),$$

where

$$\|\varphi\|_v ([\mathbf{a}] = \frac{|\varphi(\mathbf{a})|_v}{\|\mathbf{a}\|_v^d}.$$

For our purposes, we thus may replace the distance d_v by the function $\max_{\varphi\in\Phi} \|\varphi\|_v$. It follows from the product formula that when $\mathbf{a}$ runs over all elements of F^n not in $\tilde{Z}$, then

$$\prod_{w\in \mathrm{Val}(F)} \max_{\varphi\in\Phi} |\varphi(\mathbf{a})|_w \geqslant 1.$$

When $\mathbf{a}$ belongs to $\mathfrak{d}$ and w is a finite place of F then $|\varphi(\mathbf{a})|_w$ is bounded from above, by a constant c_w which may be chosen equal to 1 for almost all w. Since S contains all archimedean places of F,

$$\prod_{v\in S} \max_{\varphi\in\Phi} |\varphi(\mathbf{a})|_v \geqslant c_1 = 1/\prod_{w\notin S} c_w.$$

Consequently, for all $\mathbf{a} \in \mathfrak{d}$ outside $\tilde{Z}$,

$$\prod_{v\in S} \max_{\varphi\in\Phi} \|\varphi\|_v ([\mathbf{a}]) = \prod_{v\in S} \max_{\varphi\in\Phi} \frac{|\varphi(\mathbf{a})|_v}{\|\mathbf{a}\|_v^d} \geqslant \frac{c_1}{\prod_{v\in S} \|\mathbf{a}\|_v^d} \geqslant \frac{c_2}{\prod_{v|\infty} \|\mathbf{a}\|_v^d},$$

since, for any finite place $v \in S$, $\|\mathbf{a}\|_v$ is bounded from above on the lattice $\mathfrak{o}_F$. For any archimedean place $v \in F$, let $e_v = 1$ if $F_v = \mathbf{R}$ and $e_v = 2$ if $F_v = \mathbf{C}$. Then, using the inequality between geometric and arithmetic means, we have

$$\prod_{v|\infty} \|\mathbf{a}\|_v = 1 \cdot \prod_{v|\infty} \left(\|\mathbf{a}\|_v^{1/e_v}\right)^{e_v} \leqslant \left(\frac{1 + \sum_{v|\infty} e_v \|\mathbf{a}\|_v^{1/e_v}}{1 + [F : \mathbf{Q}]}\right)^{1+[F:\mathbf{Q}]}.$$

Now, given the equivalence of norms on the real vector space $F^n \otimes_{\mathbf{Q}} \mathbf{R}$ we may assume that $\|\cdot\|_\infty = \sum_{v|\infty} e_v \|\mathbf{a}\|_v^{1/e_v}$. This implies that there exists a positive real number c_3 such that

$$\prod_{v|\infty} \|\mathbf{a}\|_v \leqslant c_3(1 + \|\mathbf{a}\|_\infty)^{1+[F:\mathbf{Q}]},$$

and finally

$$\prod_{v\in S}\max_{\varphi\in\Phi}\|\varphi\|_v\left([\mathbf{a}]\right)\geqslant(c_2/c_3)\left(1+\|\mathbf{a}\|_\infty\right)^{-\kappa},$$

with $\kappa=d(1+[F:\mathbf{Q}])$. The lemma is proved. □

Let us fix a stratum P. Let $\mathbf{A}$ be the set of all families $A=(A_v)_{v\in S}$, where, for each $v\in S$, A_v is a maximal subset of $\mathscr{A}$ such that $D_{A_v}(F_v)\neq\varnothing$, and $d_\alpha=0$ for all $\alpha\in A_v$ on the stratum P. (Recall that by construction, d_α is constant on each stratum.) By Proposition 3.4.2 and Corollary 3.3.7, combined with the results of Section 3.3.3, for each family $A=(A_v)$ in $\mathbf{A}$ and each $\mathbf{a}$ in the stratum P, there exists a holomorphic function $\varphi_A(\mathbf{a};\cdot)$ on the half-plane $\mathrm{Re}(s)>1-\delta$ such that

$$\hat{H}(\mathbf{a};s\lambda)=\sum_{A\in\mathbf{A}}\varphi_A(\mathbf{a};s)\prod_{\alpha\in\mathscr{A}_0^D(\mathbf{a})}\zeta^S_{F_\alpha}(1+\lambda_\alpha s-\rho_\alpha)\prod_{v\in S}\prod_{\tilde\alpha\in A_v}\zeta_{F_{\tilde\alpha}}(\lambda_\alpha s-\rho_\alpha).$$

Moreover, this function $\varphi_A(\mathbf{a};s)$ satisfies estimates of the form

$$|\varphi_A(\mathbf{a};s)|\ll(1+\|a\|_\infty)^\varepsilon(1+|s|)^{\kappa+\sum\kappa'_v}\prod_{v\in S}d_v([\mathbf{a}],\partial P)^{-\kappa}\prod_{v\in S}\|\mathbf{a}\|^{-\kappa'_v},$$

where κ and κ'_v are positive real numbers, and κ'_v can be chosen arbitrarily large if v is archimedean. The series Z_P decomposes as a sum, for $A\in\mathbf{A}$, of subseries:

$$\mathrm{Z}_P(s)=\prod_{\alpha\in\mathscr{A}_0^D(\mathbf{a})}\zeta^S_{F_\alpha}(1+\lambda_\alpha s-\rho_\alpha)\prod_{v\in S}\prod_{\tilde\alpha\in A_v}\zeta_{F_{\tilde\alpha}}(\lambda_\alpha s-\rho_\alpha)\mathrm{Z}_{P,A}(s),$$

where

$$\mathrm{Z}_{P,A}(s)=\sum_{\mathbf{a}\in P}\varphi_A(\mathbf{a};s).$$

Taking κ'_v large enough for v archimedean, Lemma 3.5.2 implies that the series $\mathrm{Z}_{P,A}$ is bounded term by term by the convergent series

$$\sum_u\frac{1}{(1+\|u\|)^N},$$

where u runs over a lattice in the vector space $F^n\otimes_{\mathbf{Q}}\mathbf{R}$ and N is an arbitrarily large integer, at a cost of $(1+|s|)^N$. This provides the desired meromorphic continuation of Z_P to $\mathrm{Re}(s)>1-\delta$ and, consequently, of the height zeta function Z.

3.5.3. Leading poles (log-anticanonical line bundle). — For any character $\mathbf{a}$, let $b_{\mathbf{a}}$ be the order of the pole of the subseries in the Fourier expansion of the height zeta function corresponding to characters collinear to $\mathbf{a}$.

The order of the pole of the term corresponding to the trivial character is

$$b_0=\#(\mathscr{A}\setminus\mathscr{A}_D)+\sum_{v\in S}\max_{\substack{B\subset\mathscr{A}_{D,v}\\ D_B(F_v)\neq\varnothing}}\#B,$$

while, for $\mathbf{a} \neq 0$, one has

$$b_{\mathbf{a}} \leqslant \#\{\alpha \in (\mathscr{A} \setminus \mathscr{A}_D)\,;\, d_\alpha(\mathbf{a}) = 0\} + \sum_{v \in S} \max_{\substack{B \subset \mathscr{A}_{D,v} \\ D_B(F_v) \neq \varnothing}} \#\{\alpha \in B\,;\, d_\alpha(\mathbf{a}) = 0\}.$$

Lemma 3.5.4. — *For any nonzero character* $\mathbf{a}$, $b_{\mathbf{a}} < b_0$.

Proof. — By contradiction. Assume that $b_{\mathbf{a}} = b_0$. Comparing the formulae for $b_{\mathbf{a}}$ and b_0, we see that

a) $d_\alpha(\mathbf{a}) = 0$ for any $\alpha \in \mathscr{A}_D$

b) for any $v \in S$, there exists a subset $B \subset \mathscr{A}_{D,v}$ of maximal cardinality such that $D_B(F_v) \neq \varnothing$; moreover, $d_\alpha(\mathbf{a}) = 0$ for all $\alpha \in B$.

Fix a $\mathbf{y} \in G(F)$ such that $\langle \mathbf{a}, \mathbf{y} \rangle = 1$. Let us fix a $v \in S$ and let B be a subset as in Condition b). By definition, the rational function $f_{\mathbf{a}} = \langle \mathbf{a}, \cdot \rangle$ is defined and nonzero at the generic point of the stratum D_B. Moreover, there exists such a generic point $\mathbf{x}$ in $D_B(F_v)$, by the choice of B. Let $\mathbf{x}_\infty = \lim_{t \to \infty} t\mathbf{y} \cdot \mathbf{x}$; this is a point of $D_B(F_v)$. The rational function $t \mapsto f_{\mathbf{a}}(t\mathbf{y} \cdot \mathbf{x})$ is well defined on $\mathbf{P}^1$, and $f_{\mathbf{a}}(t\mathbf{y} \cdot \mathbf{x}) \to \infty$ when $t \to \infty$. Consequently, the limiting point $\mathbf{x}_\infty$ belongs to a divisor D_α such that $d_\alpha(\mathbf{a}) > 0$. By condition a), $\alpha \notin \mathscr{A}_D$, hence the stratum $B' = B \cup \{\alpha\}$ is contained in $D = X \setminus U$, violating condition b). □

3.5.5. Application of a Tauberian theorem. — We have shown that the height zeta function Z admits a meromorphic continuation to $\mathrm{Re}(s) > 1 - \delta$. The poles are all on the line $\mathrm{Re}(s) = 1$; with the exception of the pole at $s = 1$, they are all given by the local factor at finite places in S of the zeta function of the fields F_α, and with order

$$\sum_{\substack{v \in S \\ v \text{ finite}}} \max_{\substack{B \subset \mathscr{A}_{D,v} \\ D_B(F_v) \neq \varnothing}} \# B.$$

At $s = 1$, there may be a supplementary pole caused by the local factors at archimedean places, and by the Euler product at places outside S.

By the preceding lemma, Z has a pole of highest order at $s = 1$, contributed by the Fourier transform at the trivial character only. Consequently,

$$\lim_{s \to 1} (s-1)^{b_0} \mathrm{Z}(s) = \lim_{s \to 1} (s-1)^{b_0} \hat{H}(0; s\lambda).$$

Let us write $\hat{H}^S(0; s\lambda) = \prod_{v \notin S} \hat{H}_v(0; s\lambda)$. According to [4], Prop. 4.4.4,

$$\lim_{s \to 1} (s-1)^{\operatorname{rank} \operatorname{Pic}(U)} \hat{H}^S(0; s\lambda) = \prod_{\alpha \notin \mathscr{A}_D} \frac{1}{\rho_\alpha} \cdot \int_{U(\mathbb{A}_F^S)} \prod_{v \notin S} \delta_v(\mathbf{x}) \cdot \mathrm{d}\tau^S_{(X,D)}(\mathbf{x}).$$

Observe that the integral in this formula is the volume, with respect to the measure $\tau^S_{(X,D)}$, of the set of S-adelic integral points in $X(\mathbb{A}_F^S)$.

Moreover, for any place $v \in S$, Theorem 4.1.4 of [4] shows that

$$\lim_{s\to 1}(s-1)^{1+\dim \mathscr{C}^{\mathrm{an}}_{F_v}(D)} = \sum_{\substack{A\subset \mathscr{C}^{\mathrm{an}}_{F_v}(D)\\ \dim A=\dim \mathscr{C}^{\mathrm{an}}_{F_v}(D)}} \prod_{\alpha\in A}\frac{1}{\rho_\alpha-1}\int_{D_A(F_v)}\prod_{\alpha\in\mathscr{A}_D\setminus A}\|\mathsf{f}_\alpha\|_v(\mathbf{x})^{-1}\,\mathrm{d}\tau_{D_A}(\mathbf{x}).$$

We have

$$b_0 = \operatorname{rank}\operatorname{Pic}(U) + \sum_{v\in S}(1+\dim \mathscr{C}^{\mathrm{an}}_{F_v}(D)).$$

Other poles of $\hat{H}(0;\lambda s)$ on the line $\operatorname{Re}(s)=1$ are contained in finitely many arithmetic progressions, contributed by finitely many local factors of Dedekind zeta functions for $v\in S$. Combining local and global estimates in vertical strips, as in Sections 4.1 and 4.3 of [4], and applying the Tauberian theorem A.15 of that reference, we finally obtain our Main theorem:

Theorem 3.5.6. — *Let X be a smooth projective equivariant compactifications of a vector group $G=\mathbb{G}_a^n$ over a number field F. Assume that $X\setminus G$ is geometrically a strict normal crossing divisor.*

Let $U\subset X$ a dense G-invariant open subset and let $D=X\setminus U$ be the boundary. Let a smooth adelic metric be given on the log-anticanonical divisor $-(K_X+D)$. Let S be a finite set of places of F containing the archimedean places.

Let $\mathscr{U}(\mathfrak{o}_{F,S})$ be the set of S-integral points on an integral model $\mathscr{U}$ of U as in Section 3.1.8. Let $N(B)$ be the number of points in $G(F)\cap\mathscr{U}(\mathfrak{o}_{F,S})$ of log-anticanonical height $\leqslant B$, as in Section 3.1.2. Let

$$b = \operatorname{rank}\operatorname{EP}(U) + \sum_{v\in S}(1+\dim \mathscr{C}^{\mathrm{an}}_{F_v}(D))$$

Then, as $B\to\infty$,

$$N(B)\sim\left(\prod_{\alpha\notin\mathscr{A}_D}\frac{1}{\rho_\alpha}\cdot\tau^S_{(X,D)}(U(\mathbb{A}_F^S)^{\mathrm{int}})\cdot\prod_{v\in S}\tau_v^{\max}(D(F_v))\right)B(\log B)^{b-1},$$

where

$$\tau^S_{(X,D)}(U(\mathbb{A}_F^S)^{\mathrm{int}}) = \int_{U(\mathbb{A}_F^S)}\prod_{v\notin S}\delta_v(\mathbf{x})\cdot\mathrm{d}\tau^S_{(X,D)}(\mathbf{x})$$

while, for each $v\in S$, $\tau_v^{\max}$ is the measure on $D(F_v)$ given by

$$\mathrm{d}\tau_v^{\max}(x) = \sum_{\substack{A\subset \mathscr{C}^{\mathrm{an}}_{F_v}(D)\\ \dim A=\dim \mathscr{C}^{\mathrm{an}}_{F_v}(D)}} \prod_{\alpha\in A}\frac{1}{\rho_\alpha-1}\prod_{\alpha\in\mathscr{A}_{D,v}\setminus A}\|\mathsf{f}_\alpha\|_v(\mathbf{x})^{-1}\,\mathrm{d}\tau_{D_A}(\mathbf{x}).$$

Remark 3.5.7. — Let $V(B)$ be the volume, with respect to the chosen Haar measure on $G(\mathbb{A}_F)$ of the set of S-integral adelic points of log-anticanonical height $\leqslant B$. By

definition, $\hat{H}(0; s\lambda)$ is the Stieltjes–Mellin transform of the measure $\mathrm{d}V(B)$ on $\mathbf{R}_+$. By the same Tauberian theorem, one has

$$N(B) \sim V(B).$$

3.5.8. Equidistribution. — Note that all considerations above apply to an arbitrary smooth adelic metrization of the log-anticanonical line bundle.

Applying the abstract equidistribution theorem (Prop. 2.5.1) of [4], we obtain that integral points of height $\leqslant B$ equidistribute, when $B \to \infty$, towards the unique probability measure proportional to

$$\prod_{v \notin S} \delta_v(x) \cdot \mathrm{d}\tau^S_{(X,D)}(x) \cdot \prod_{v \in S} \mathrm{d}\tau_v^{\max}(x).$$

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